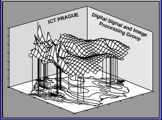

MR IMAGE COMPRESSION BY HAAR WAVELET TRANSFORM

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ICT PRAGUE

Process Control 2007, Štrbské Pleso

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Digital signal processing tasks

- Noise reduction.
- Feature extraction ⇒ classification.
- Restoration of missing or corrupted components.
- Image compression and coding.

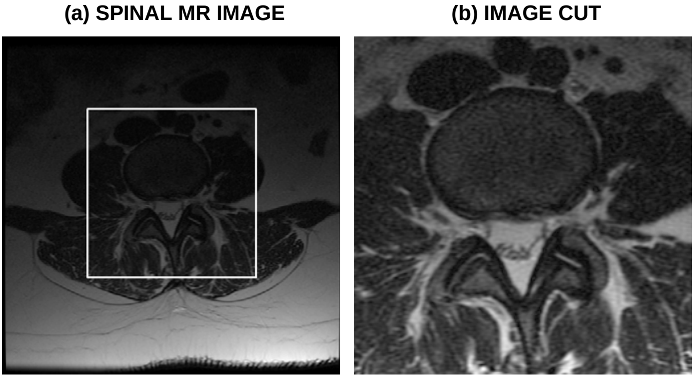
Possible solution - the discrete wavelet transform (DWT).

The Haar Transform

- The Haar function ⇒ the Haar transform (HT).
- Simple computation algorithm.
- Orthonormal transform.
- Satisfies the perfect reconstruction conditions.

Introduction

Axial spine MR image and its cut



- Application of the HT to magnetic resonance (MR) images.

1-D Haar Transform

One-level decomposition

Let us have a signal $\{x(n)\}_{n=0}^{N-1}$. For $n = 0, 2, \ldots, N-2$:

$$\begin{pmatrix} X_n \\ X_{n+1} \end{pmatrix} = \mathbf{T} \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} \tag{1}$$

Decomposition matrix

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2}$$

Resulting sequence

- $\{X_0, X_2, \ldots, X_{N-2}\}$ - low-pass decomposition values.
- $\{X_1, X_3, \ldots, X_{N-1}\}$ - high-pass sequence.

2-D Haar Transform

One-level decomposition

Image $[g(n,m)]_{N,M}$. For $n=0,2,\ldots,N-2,\ m=0,2,\ldots,M-2$:

- Column-wise decomposition:

$$\begin{pmatrix} G1_{n,m} & G1_{n,m+1} \\ G1_{n+1,m} & G1_{n+1,m+1} \end{pmatrix} = \mathbf{T} \begin{pmatrix} g_{n,m} & g_{n,m+1} \\ g_{n+1,m} & g_{n+1,m+1} \end{pmatrix} \tag{3}$$

- Row-wise decomposition:

$$\begin{pmatrix} G_{n,m} & G_{n,m+1} \\ G_{n+1,m} & G_{n+1,m+1} \end{pmatrix} = \begin{pmatrix} G1_{n,m} & G1_{n,m+1} \\ G1_{n+1,m} & G1_{n+1,m+1} \end{pmatrix} \mathbf{T}^T \tag{4}$$

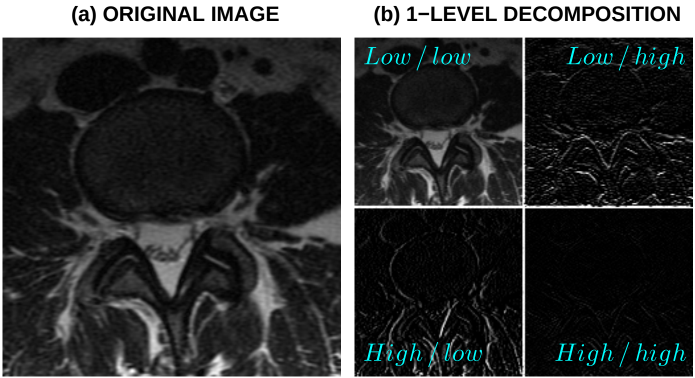
Resulting matrix

- The low/low-pass submatrix:

$$\begin{pmatrix} G_{0,0} & G_{0,2} & \cdots & G_{0,M-2} \\ G_{2,0} & G_{2,2} & \cdots & G_{2,M-2} \\ & & \cdots & \\ G_{N-2,0} & G_{N-2,2} & \cdots & G_{N-2,M-2} \end{pmatrix}$$

2-D Haar Transform

One-level MR image decomposition



Orthonormality

Orthonormal matrix

A real-valued matrix $\mathbf{A}$ of size $N \times N$ is orthonormal, if

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}\,\mathbf{A}^{T} = \mathbf{I_N} \tag{5}$$

$\mathbf{I_N}$ ... identity matrix of size $N \times N$.

Analysis

$$\mathbf{w} = \mathbf{A}\,\mathbf{x} \tag{6}$$

$\mathbf{x}$ ... signal $\mathbf{x}$ of size $N \times 1$.
$\mathbf{w}$ ... coefficients vector of size $N \times 1$.

Synthesis

$$\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{w} = \mathbf{A}^{T}\,\mathbf{w} \tag{7}$$

Parseval's theorem

Signal energy

$$\varepsilon_x = \|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T \mathbf{x} = \sum_{n=0}^{N-1} x_n^2 \qquad (8)$$

ε_x . . . total energy in a discrete signal $\mathbf{x}$

Preserving the energy

$$\begin{aligned} \varepsilon_w &= \|\mathbf{w}\|^2 = \mathbf{w}^T \mathbf{w} = (\mathbf{A}\,\mathbf{x})^T \mathbf{A}\,\mathbf{x} = \\ &= \mathbf{x}^T \mathbf{A}^T \mathbf{A}\,\mathbf{x} = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|^2 = \varepsilon_x \end{aligned} \qquad (9)$$

$\mathbf{w}$ . . . orthonormal transform (matrix $\mathbf{A}$) coefficients vector.
ε_w . . . total energy of these coefficients.

Parseval's theorem

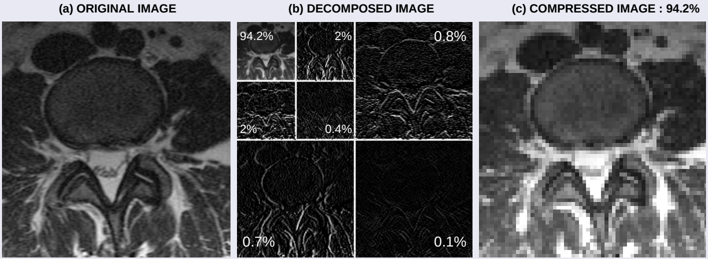
$$\varepsilon_x = \sum_{n=0}^{N-1} x_n^2 = \sum_{k=0}^{N-1} w_k^2 \qquad (10)$$

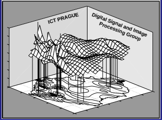
Parseval's theorem

MR image compression

- Parseval's theorem $\Rightarrow$ the proportion of energy in each HT coefficients set $\Rightarrow$ entropy $\Rightarrow$ the extent of compression ($\downarrow$ bits per pixel).

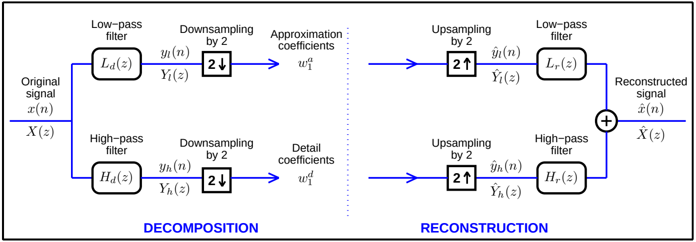
Proportions of energy





Decomposition

Subband coding algorithm



Filtering

- $y_l(n) = \sum_{k=0}^{M-1} l_d(k)x(n-k) \xrightarrow{Z\text{-tr.}} Y_l(z) = L_d(z)X(z)$
- $y_h(n) = \sum_{k=0}^{M-1} h_d(k)x(n-k) \xrightarrow{Z\text{-tr.}} Y_h(z) = H_d(z)X(z)$

M . . . filter order.

Decomposition

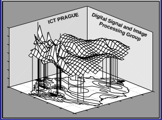
Downsampling by 2

- Low-pass sequence - approximations:

$$w_1^a = \{y_l(0), y_l(2), y_l(4), ..., y_l(N-2)\}$$

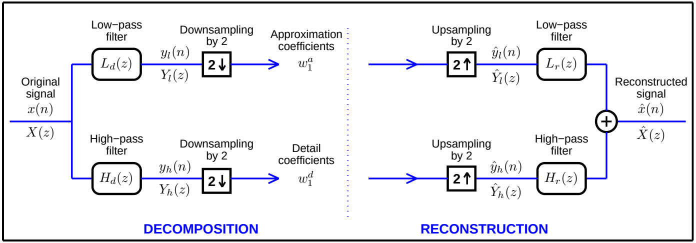
- High-pass sequence - details:

$$w_1^d = \{y_h(0), y_h(2), y_h(4), ..., y_h(N-2)\}$$



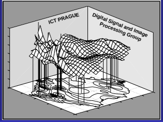
Reconstruction

Subband coding algorithm



Upsampling by 2

- $\widehat{y}_l(n) = \{y_l(0), 0, y_l(2), 0, \ldots, y_l(N-2), 0\}$
- $\widehat{y}_h(n) = \{y_h(0), 0, y_h(2), 0, \ldots, y_h(N-2), 0\}$



Reconstruction

Z-transform

$$\widehat{Y}_l(z) = \sum_{n=0}^{\infty} \widehat{y}_l(n) z^{-n} = \frac{1}{2}[Y_l(z) + Y_l(-z)] \quad (11)$$

$$\widehat{Y}_h(z) = \sum_{n=0}^{\infty} \widehat{y}_h(n) z^{-n} = \frac{1}{2}[Y_h(z) + Y_h(-z)] \quad (12)$$

Filtering

$$\widehat{X}_l(z) = \frac{1}{2} L_r(z)[Y_l(z) + Y_l(-z)] \quad (13)$$

$$\widehat{X}_h(z) = \frac{1}{2} H_r(z)[Y_h(z) + Y_h(-z)] \quad (14)$$

Summation

$$\widehat{X}(z) = \widehat{X}_l(z) + \widehat{X}_h(z) =$$

$$= \frac{1}{2} L_r(z)[Y_l(z)+Y_l(-z)] + \frac{1}{2} H_r(z)[Y_h(z)+Y_h(-z)] \quad (15)$$

Perfect Reconstruction (PR) Conditions

Substituting for $Y_l(z)$ and $Y_h(z)$

$$\begin{aligned}\widehat{X}(z) &= \frac{1}{2}X(z)[L_r(z)L_d(z)+H_r(z)H_d(z)] + \\ &+\frac{1}{2}X(-z)[L_r(z)L_d(-z)+H_r(z)H_d(-z)]\end{aligned} \tag{16}$$

Perfect reconstruction requirement

$$\widehat{X}(z) \equiv X(z) \tag{17}$$

Perfect reconstruction (PR) conditions

$$L_r(z)L_d(z) + H_r(z)H_d(z) \equiv 2 \tag{18}$$

$$L_r(z)L_d(-z) + H_r(z)H_d(-z) \equiv 0 \tag{19}$$

- Eq. (19) - anti-aliasing condition

Perfect Reconstruction (PR) Conditions

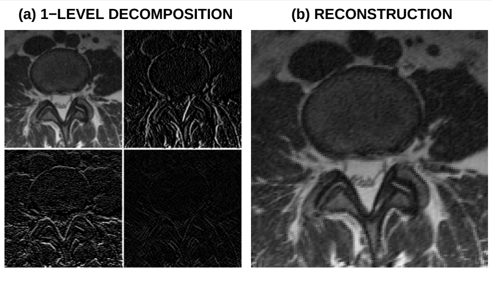
HT filter bank

- Decomposition filters:

  Matrix $\mathbf{T} \rightarrow l_d = 1/\sqrt{2}\,[1, 1]$, $h_d = 1/\sqrt{2}\,[1, -1]$.
- Reconstruction filters:

  $\mathbf{T}^{-1} = \mathbf{T}^T = \mathbf{T} \rightarrow l_r = l_d$ and $h_r = h_d$, except the inverse time course.

Reconstruction from the HT coefficients



Conclusions

Haar transform

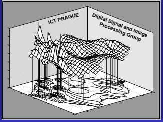
- Simple and useful tool for image compression.
- Computation algorithm.

Orthonormality

- Description of orthonormal discrete transforms.
- $\Rightarrow$ Parseval's theorem $\Rightarrow$ energy in the coefficients $\Rightarrow$ extend of image compression.
- $\Rightarrow$ Perfect reconstruction.

PR conditions

- Derivation of the perfect reconstruction (PR) conditions using the Z-transform.



Further Reading

- D. B. Percival and A. T. Walden.
  Wavelet Methods for Time Series Analysis.
  Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2000.
- G. Menegaz and L. Grewe and J. P. Thiran.
  Multirate Coding of 3D Medical Data.
  Proceedings of the 2000 International Conference on Image Processing. volume 3, pages 656 – 659, IEEE, 2000.
- J. Wang and H. K. Huang.
  Medical Image Compression by Using Three-Dimensional Wavelet Transform.
  IEEE Transactions on Medical Imaging. volume 15(4):547 – 554, 1996.
- N. Kingsbury.
  4F8 Image Coding Course.
  http://cnx.org/, June 2006.