FIR APPROXIMATION OF ND LOG FILTER IN FOURIER DOMAIN

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Abstract

The Laplacian of Gaussian (LoG) filter is a kind of linear and infinite impulse (IIR) high pass filter, which is a useful tool for the edge detection in the n-dimensional space. It is also called Marr-Hildreth filter, being serious and reputable model of human contextual vision in 2D. Applying n-dimensional Fourier transform, the transcendent but smooth transfer function is obtained as $\mathrm{F}(\boldsymbol{\omega}) = \omega^2 \exp(-\sigma^2\omega^2/2)$ where $\sigma > 0$ is a space width parameter and $\omega = \|\boldsymbol{\omega}\|_2$, $\boldsymbol{\omega} \in \mathcal{R}^n$ for given dimension $n \in \mathcal{N}$. The transfer function is radial and thus symmetric in the frequency coordinates. The aim of our study is to approximate the nD LoG filter as a finite impulse response (FIR) filter of size $r \in \mathcal{N}$ consisting of $(2r+1)^n$ non-zero elements at most. The Fourier transform of symmetric FIR is also symmetric function but not radial, of course. The analytic investigation of LoG and FIR transfer functions (at the neighborhood of $\boldsymbol{\omega}$=0) enables to define the degree of FIR radiality and the degree of FIR approximation. The Matlab symbolic toolbox was used to design adequate FIR approximations of various dimension, size, radiality and the other approximation characteristics.

1 LoG filter and its properties

LoG - Laplacian of Gaussian is well known IIR filter. N-dimensional case can be written as

$$\mathrm{F}(\boldsymbol{\omega}) = \omega^2 \exp(-\sigma^2\omega^2/2), \tag{1}$$

σ is width parameter and it should be grater than 0 and $\omega = \|\boldsymbol{\omega}\|_2$, $\boldsymbol{\omega} \in \mathcal{R}^n$ for given dimension $n \in \mathcal{N}$. LoG was created as a combination of Laplacian and Gaussian and has properties of both operators. Laplacian itself is an approximation of second derivation and so high-pass filter is used for edge detection very often. Transfer function of Gaussian

$$\mathrm{H}(\boldsymbol{\omega}) = \exp(-\sigma^2\omega^2/2). \tag{2}$$

is reversely low-pass filter for de-noising purposes.

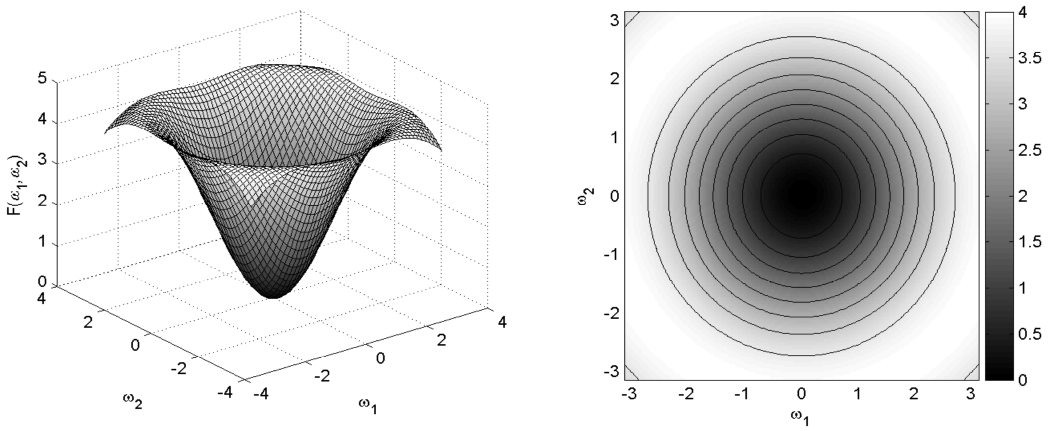

Figure 1: Spectrum of LoG with $\sigma^2 = \frac{1}{6}$

It is important to further work that Laplacian of Gaussian is band-pass a filter which punctuates higher frequencies then other. Radiality is another important property. Transfer function of LoG is a radial function of any order. It means that it satisfies equation

$$\mathrm{H}(\boldsymbol{\omega}) = \sum_{k=0}^{m} h_k \omega^{2k} + \mathrm{R}_{2m+2}(\boldsymbol{\omega}) \tag{3}$$

for all $m \in \mathcal{N}$ and has no residual function R. $\omega = \|\boldsymbol{\omega}\|_2 = \sqrt{\sum_{l=1}^{n} \omega_l^2}$, $n \in \mathcal{N}$ is a dimension and h_k is a coefficient. Residual function is a part of function $\mathrm{G}(\boldsymbol{\omega})$ which cannot be expressed as the distance from the beginning of coordinate system in this context. Practical impact of radiality is that LoG has circle contour lines, as you can see on figure 15. Radiality also makes Laplacian of Gaussian swap invariant and symmetric.

It should also be mentioned that Laplacian of Gaussian is a model of human vision as Marr theory supposes. This fact is not really important for this work.

2 FIR approximation of LoG

Let's suppose multi-dimensional Fourier series

$$\mathrm{G}(\boldsymbol{\omega}) = \prod_{l=1}^{n} \sum_{k_l=-r}^{r} a_{k_l} \exp(2\pi \mathrm{i} k_l \omega_l). \tag{4}$$

It is possible to simplify it to

$$\mathrm{G}(\boldsymbol{\omega}) = \prod_{l=1}^{n} \sum_{k_l=-r}^{r} a_{k_l} \cos(k_l \omega_l) \tag{5}$$

using Euler formula. Symbol r is order of approximation and n dimension. Functions $\sin(k_l\omega_l)$ are not needed because Laplacian of Gaussian is a radial function. Specially for two- (resp. three-) dimensional case we obtain (6) ((7) respectively).

$$\begin{aligned} \mathrm{G}(\omega_1, \omega_2) = a_0a_0 + a_1a_0\cos(\omega_1) + a_{-1}a_0\cos(\omega_1) + a_0a_1\cos(\omega_2) + a_0a_{-1}\cos(\omega_2) + \\ +(a_1a_1 + a_{-1}a_1 + a_1a_{-1} + a_{-1}a_{-1})\cos(\omega_1)\cos(\omega_2) + \ldots \end{aligned} \tag{6}$$

$$\begin{gathered} \mathrm{G}(\omega_1, \omega_2, \omega_3) = a_0a_0a_0 + (a_1a_0a_0 + a_{-1}a_0a_0)\cos(\omega_1) + \\ +(a_0a_1a_0 + a_0a_{-1}a_0)\cos(\omega_2) + (a_0a_0a_1 + a_0a_0a_{-1})\cos(\omega_3) + \\ +(a_1a_1a_0 + a_1a_{-1}a_0 + a_{-1}a_1a_0 + a_{-1}a_{-1}a_0)\cos(\omega_1)\cos(\omega_2) + \\ +(a_1a_0a_1 + a_1a_0a_{-1} + a_{-1}a_0a_1 + a_{-1}a_0a_{-1})\cos(\omega_1)\cos(\omega_3) + \\ +(a_0a_1a_1 + a_0a_1a_{-1} + a_0a_{-1}a_1 + a_0a_{-1}a_{-1})\cos(\omega_2)\cos(\omega_3) + \\ +(a_1a_1a_1 + a_{-1}a_1a_1 + a_1a_{-1}a_1 + a_1a_1a_{-1} + a_{-1}a_{-1}a_1 + \\ +a_{-1}a_1a_{-1} + a_1a_{-1}a_{-1} + a_{-1}a_{-1}a_{-1})\cos(\omega_1)\cos(\omega_2)\cos(\omega_3) + \ldots \end{gathered} \tag{7}$$

Now let's consider substitution of bilinear or trilinear coefficients, they were created by multiplication of one-dimensional Fourier series (like the following one).

$$\begin{bmatrix} a_1a_{-1} & a_1a_0 & a_1a_1 \\ a_0a_{-1} & a_0a_0 & a_0a_1 \\ a_{-1}a_{-1} & a_{-1}a_0 & a_{-1}a_1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1a_1 & a_1a_0 & a_1a_1 \\ a_0a_1 & a_0a_0 & a_0a_1 \\ a_1a_1 & a_1a_0 & a_1a_1 \end{bmatrix} \Rightarrow \begin{bmatrix} c & b & c \\ b & a & b \\ c & b & c \end{bmatrix} \tag{8}$$

This means that index k_l could be understood as distance ($a_{k_l} = a_{-k_l}$) and all permutations of each combination with repetition of coefficients a with indexes $|k_l|$ are substituted with the same variable. Substitution is needed because we don't want to create separable edge detecting operators (it is not even possible) and feasible because LoG is a radial function. Hence (6) and (7) can be rewritten as

$$\mathrm{G}(\omega_1, \omega_2) = a + 2b(\cos(\omega_1) + \cos(\omega_2)) + 4c\cos(\omega_1)\cos(\omega_2) + \ldots \tag{9}$$

$$\begin{gathered}\mathrm{G}(\omega_1,\omega_2,\omega_3)=a+2b(\cos(\omega_1)+\cos(\omega_2)+\cos(\omega_3))\\ +4c(\cos(\omega_1)\cos(\omega_2)+\cos(\omega_1)\cos(\omega_3)+\cos(\omega_2)\cos(\omega_3))+\\ +8d\cos(\omega_1)\cos(\omega_2)\cos(\omega_3)+\dots\end{gathered}\tag{10}$$

Number of combinations can be evaluated as $\binom{n+r-1}{n}$. Meaning of the symbols is the same as in the previous equations.

Determination of coefficients should be done with some care but still a lot of approaches exist. We decided to use approximation in point $\boldsymbol{\omega}=\mathbf{0}$ with respect to radiality. Results have very similar shape to LoG even for low order of approximation thus FIR filters become high-pass.

Definition of radial function provides very important information about values of derivation of Laplacian of Gaussian (function $\mathrm{F}(\boldsymbol{\omega})$) in point $\boldsymbol{\omega}=\mathbf{0}$. It says that only derivation of the type

$$\frac{\partial^{2s+2t}\mathrm{F}}{\partial\omega_1^{2s}\partial\omega_2^{2t}}\neq 0\tag{11}$$

or

$$\frac{\partial^{2s+2t+2u}\mathrm{F}}{\partial\omega_1^{2s}\partial\omega_2^{2t}\partial\omega_3^{2u}}\neq 0\tag{12}$$

are non-zero in two-dimensional case (or 3D respectively), $s,t,u\in\mathcal{N}_0$. Other important property comes from symmetry and it is

$$\frac{\partial^{2s+2t}\mathrm{F}}{\partial\omega_1^{2s}\partial\omega_2^{2t}}=\frac{\partial^{2s+2t}\mathrm{F}}{\partial\omega_1^{2t}\partial\omega_2^{2u}}\tag{13}$$

and

$$\begin{gathered}\frac{\partial^{2s+2t+2u}\mathrm{F}}{\partial\omega_1^{2s}\partial\omega_2^{2t}\partial\omega_3^{2u}}=\frac{\partial^{2s+2t+2u}\mathrm{F}}{\partial\omega_1^{2s}\partial\omega_2^{2u}\partial\omega_3^{2t}}=\frac{\partial^{2s+2t+2u}\mathrm{F}}{\partial\omega_1^{2t}\partial\omega_2^{2s}\partial\omega_3^{2u}}=\\ =\frac{\partial^{2s+2t+2u}\mathrm{F}}{\partial\omega_1^{2t}\partial\omega_2^{2u}\partial\omega_3^{2s}}=\frac{\partial^{2s+2t+2u}\mathrm{F}}{\partial\omega_1^{2u}\partial\omega_2^{2s}\partial\omega_3^{2t}}=\frac{\partial^{2s+2t+2u}\mathrm{F}}{\partial\omega_1^{2u}\partial\omega_2^{2t}\partial\omega_3^{2s}}\end{gathered}\tag{14}$$

Approximation - in the terms of this text - means to put equal sign between derivation of Laplacian of Gaussian and cosine series. Definition of radiality gives an answer to other question; which values of derivations should be preserved when we want radiality of some order. It will be derivation with the same value as coefficients of binomial (trinomial) expansion in two- (three-) dimensional case like

$$\mathrm{H}(\omega_1,\omega_2)=\sum_{k=0}^{m}h_k(\omega_1^2+\omega_2^2)^k=h_0+h_1(\omega_1^2+\omega_2^2)+h_2(\omega_1^4+2\omega_1^2\omega_2^2+\omega_2^4)+\dots\tag{15}$$

respectively

$$\begin{gathered}\mathrm{H}(\omega_1,\omega_2,\omega_3)=\sum_{k=0}^{m}h_k(\omega_1^2+\omega_2^2+\omega_3^2)^k=\\ =h_0+h_1(\omega_1^2+\omega_2^2+\omega_3^3)+h_2(\omega_1^4+\omega_2^4+\omega_3^4+2\omega_1^2\omega_2^2+2\omega_1^2\omega_3^2+2\omega_2^2\omega_3^2)\end{gathered}\tag{16}$$

Now it is possible to show algorithm how to approximate LoG to FIR filter by putting all information together. We obtain a set of equations. It is possible to have equations of two type in the set.

$$\left.\frac{\partial^{\sum_{l=1}^{n}k_l}\mathrm{F}(\boldsymbol{\omega})}{\partial\omega_1^{k_1}\partial\omega_2^{k_2}\dots}\right|_{\boldsymbol{\omega}=\mathbf{0}}=\left.\frac{\partial^{\sum_{l=1}^{n}k_l}\mathrm{G}(\boldsymbol{\omega})}{\partial\omega_1^{k_1}\partial\omega_2^{k_2}\dots}\right|_{\boldsymbol{\omega}=\mathbf{0}}\tag{17}$$

where $k_1=0\dots m$, $k_2=0\dots m-k_1$ etc and r is order of approximation. Second type of equations comes from

$$\left.\frac{\partial^{\sum_{l=1}^{n}k_l}\mathrm{G}(\boldsymbol{\omega})}{\partial\omega_1^{k_1}\partial\omega_2^{k_2}\dots}\right|_{\boldsymbol{\omega}=\mathbf{0}}=\gamma\left.\frac{\partial^{\sum_{l=1}^{n}\kappa_l}\mathrm{G}(\boldsymbol{\omega})}{\partial\omega_1^{\kappa_1}\partial\omega_2^{\kappa_2}\dots}\right|_{\boldsymbol{\omega}=\mathbf{0}}\tag{18}$$

Derivations respect the same order of radiality in (15) (or (16) in three-dimensional case). Coefficient γ is a ratio of derivations and it is possible to calculate it like

$$\gamma = \left. \frac{\frac{\partial^{\sum_{l=1}^{n} k_l} \mathrm{F}(\boldsymbol{\omega})}{\partial \omega_1^{k_1} \partial \omega_2^{k_2} \ldots}}{\frac{\partial^{\sum_{l=1}^{n} \kappa_l} \mathrm{F}(\boldsymbol{\omega})}{\partial \omega_1^{\kappa_1} \partial \omega_2^{\kappa_2} \ldots}} \right|_{\boldsymbol{\omega}=\mathbf{0}} \tag{19}$$

Equation from (17) and (18) are combinable thus many of FIR filters with slightly different weights should be obtained. Example of approximation follows for $r = 2$ and $n = 2$.

$$\begin{aligned} \mathrm{G}(\omega_1, \omega_2) &= \mathrm{F}(\omega_1, \omega_2) \\ \frac{\partial^2 \mathrm{G}(\omega_1,\omega_2)}{\partial \omega_1^2} &= \frac{\partial^2 \mathrm{F}(\omega_1,\omega_2)}{\partial \omega_1^2} \\ \frac{\partial^4 \mathrm{G}(\omega_1,\omega_2)}{\partial \omega_1^2 \partial \omega_2^2} &= \frac{\partial^4 \mathrm{F}(\omega_1,\omega_2)}{\partial \omega_1^2 \partial \omega_2^2} \end{aligned} \tag{20}$$

Accomplishing of suggested operations and substitution $\omega_1 = 0$ and $\omega_2 = 2$ changes (20) to

$$\begin{aligned} a + 4b + 4c &= 0 \\ -2b - 4c &= 2 \\ 4c &= -4\sigma^2 \end{aligned} \tag{21}$$

The solution of the set of linear equation can be rewritten as a convolution mask

$$\begin{bmatrix} -\sigma^2 & 2\sigma^2 - 1 & -\sigma^2 \\ 2\sigma^2 - 1 & -4\sigma^2 + 4 & 2\sigma^2 - 1 \\ -\sigma^2 & 2\sigma^2 - 1 & -\sigma^2 \end{bmatrix} \tag{22}$$

or as transfer function

$$\mathrm{G}(\omega_1, \omega_2) = (-4\sigma^2 + 4) + 2(2\sigma^2 - 1)(\cos(\omega_1) + \cos(\omega_2)) + 4(-\sigma^2)\cos(\omega_1)\cos(\omega_2). \tag{23}$$

3 Selected FIR filters

Following convolution masks (3×3 - table 1) have format

$$\begin{bmatrix} c & b & c \\ b & a & b \\ c & b & c \end{bmatrix} \tag{24}$$

and transfer function is possible to create as

$$\mathrm{G}(\omega_1, \omega_2) = a + 2b(\cos(\omega_1) + \cos(\omega_2)) + 4c\cos(\omega_1)\cos(\omega_2). \tag{25}$$

Masks of type 5×5 (table 2)satisfy format

$$\begin{bmatrix} f & e & d & e & f \\ e & c & b & c & e \\ d & b & a & b & d \\ e & c & b & c & e \\ f & e & d & e & f \end{bmatrix} \tag{26}$$

Table 1: Two-dimensional convolution masks, 3×3

a	b	c	σ^2	*Order of Radiality*
a	$1-\frac{1}{2}a$	$-1+\frac{1}{4}a$	$1-\frac{1}{4}a$	1
a	$-\frac{1}{5}a$	$-\frac{1}{20}a$	$-1+\frac{1}{4}a$	1
20	-4	-1	$\frac{1}{6}$	2
14	-6	-1	$\frac{1}{6}$	2
36	-8	-1	$\frac{1}{10}$	2
28	-5	-2	$\frac{2}{9}$	2
52	-12	-1	$\frac{1}{14}$	2

and transfer function is

$$\begin{gathered}\mathrm{G}(\omega_1,\omega_2) = a + 2b(\cos(\omega_1)+\cos(\omega_2)) + 4c\cos(\omega_1)\cos(\omega_2)+ \\ +2d(\cos(2\omega_1)+\cos(2\omega_2)) + 4e(\cos(2\omega_1)\cos(\omega_2)+\cos(\omega_1)\cos(2\omega_2)) + 4f\cos(2\omega_1)\cos(2\omega_2)\end{gathered} \tag{27}$$

Symbol r means order of radiality.

Table 2: Two-dimensional convolution masks, 5×5

a	b	c	d	e	f
a	$-14f+\frac{3}{10}-\frac{5}{18}a$	$18f-\frac{1}{30}-\frac{1}{18}a$	$5f-\frac{1}{10}+\frac{1}{36}a$	$-5f-\frac{1}{12}+\frac{1}{36}a$	f
$\sigma^2=\frac{7}{10}+6f-\frac{1}{6}a,\ r=3$					
a	$-\frac{2}{3}a+\frac{19}{15}$	$\frac{4}{9}a-\frac{134}{105}$	$\frac{1}{6}a-\frac{187}{420}$	$-\frac{1}{9}a+\frac{11}{42}$	$\frac{1}{36}a-\frac{29}{420}$
$\sigma^2=\frac{2}{7},\ r=3$					
$576f+\frac{27}{10}$	$-174f-\frac{9}{20}$	$-14f-\frac{11}{60}$	$21f-\frac{1}{40}$	$11f-\frac{1}{120}$	f
$\sigma^2=\frac{1}{4}-90f,\ r=3$					
6228	-960	-448	-84	-32	-1
$\sigma^2=\frac{2}{7},\ r=4$					
14508	-792	-1796	-612	-212	-7
$\sigma^2=\frac{2}{5},\ r=4$					
180	-32	-12	-1	0	0
$\sigma^2=\frac{1}{5},\ r=3$					

Finally, scheme of three-dimensional convolution masks is possible to write as

$$\begin{bmatrix} d & c & d \\ c & b & c \\ d & c & d \end{bmatrix}, \quad \begin{bmatrix} c & b & c \\ b & a & b \\ c & b & c \end{bmatrix}, \quad \begin{bmatrix} d & c & d \\ c & b & c \\ d & c & d \end{bmatrix} \tag{28}$$

and transfer function as

$$\begin{gathered}\mathrm{G}(\omega_1,\omega_2,\omega_3) = a + 2b(\cos(\omega_1)+\cos(\omega_2)+\cos(\omega_3))+ \\ +4c(\cos(\omega_1)\cos(\omega_2)+\cos(\omega_1)\cos(\omega_3)+\cos(\omega_2)\cos(\omega_3)) + 8d\cos(\omega_1)\cos(\omega_2)\cos(\omega_3).\end{gathered} \tag{29}$$

Results are in table 3.

Table 3: Two-dimensional convolution masks, 3×3

a	b	c	d	σ^2	*Order of Radiality*
a	$-\frac{1}{2}a + \frac{5}{3}$	$\frac{1}{4}a - \frac{7}{6}$	$-\frac{1}{8}a + \frac{1}{2}$	$\frac{1}{6}$	2
a	$-\frac{1}{2} + \frac{9}{5}$	$\frac{1}{4}a - \frac{13}{10}$	$-\frac{1}{8} + \frac{3}{5}$	$\frac{1}{10}$	2
a	$-\frac{1}{6}a + \frac{28}{81}$	$-\frac{4}{27}$	$-\frac{1}{27}$	$\frac{2}{9}$	2
144	-16	-4	-1	$\frac{1}{6}$	2
8	6	-5	2	$\frac{1}{6}$	2
24	-2	-1	0	$\frac{1}{6}$	2
56	-8	0	1	$\frac{1}{6}$	2

4 Results

Approximation method is useable for obtaining results of different types. Focus on radiality preserves highlighting of higher frequencies in FIR filters. Derived convolution masks or transfers functions could speed up calculations because rational ratio of weights was achieved. Thus calculations with floating point are not needed. This effect could be bigger if transfer functions with zero coefficients were used.

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6 Apendix

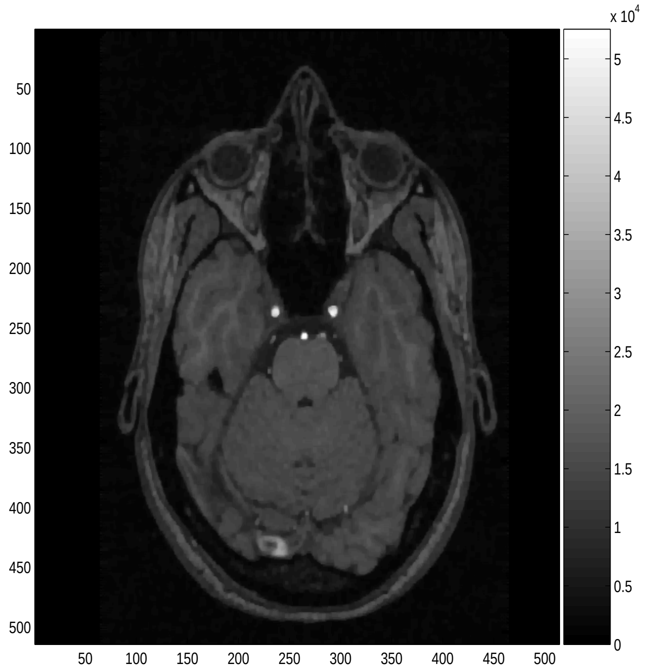

Figure 2: Original image

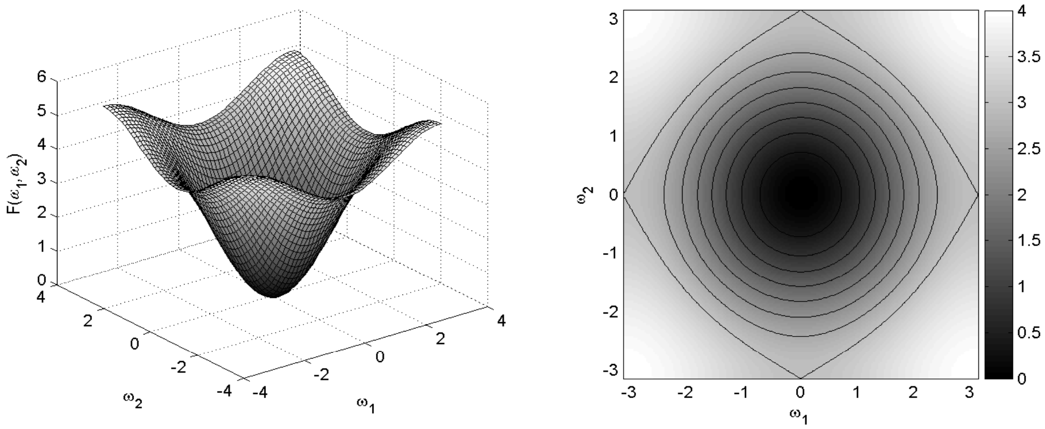

Figure 3: Spectrum of 3×3 FIR filter, $\sigma^2 = \frac{1}{6}$, $r = 2$

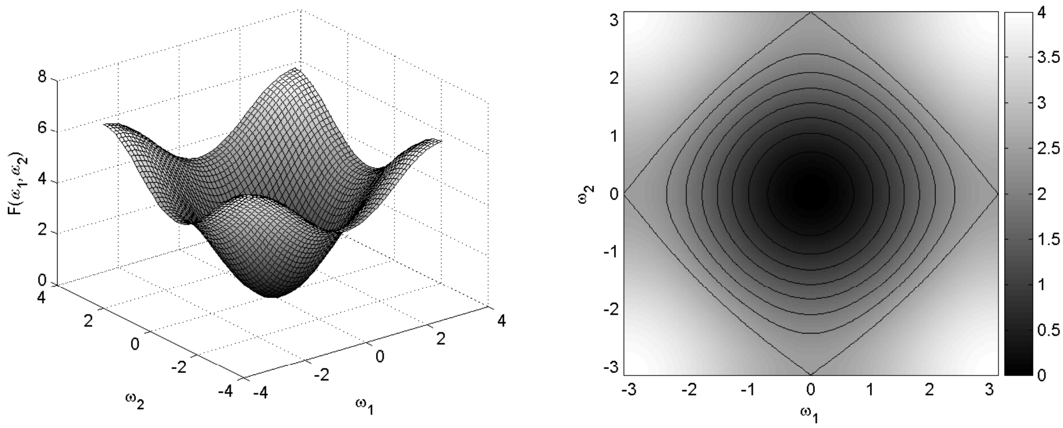

Figure 4: Spectrum of 3×3 FIR filter, $\sigma^2 = \frac{1}{10}$, $r = 2$

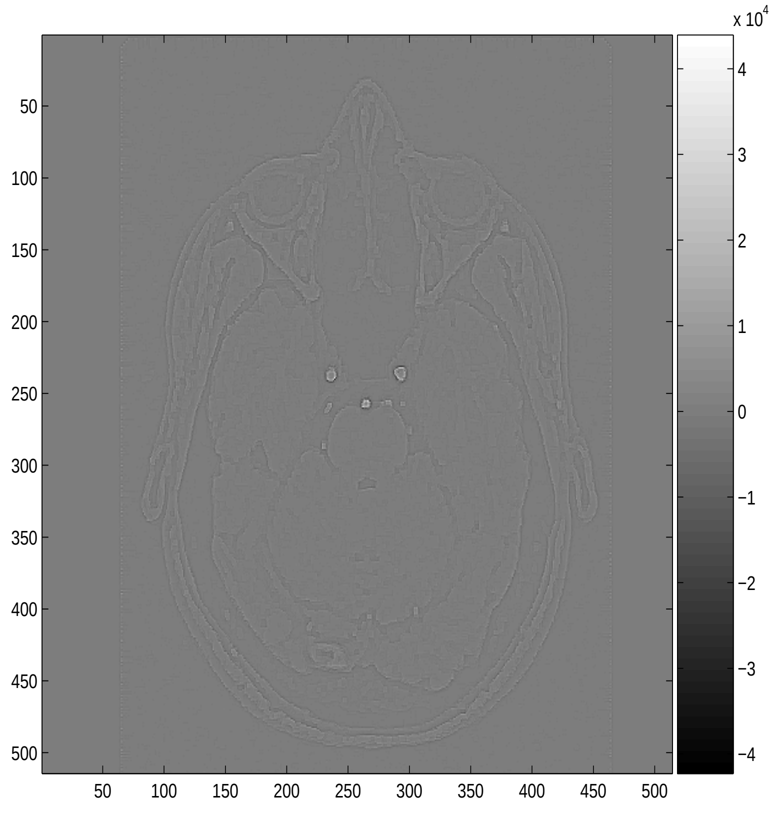

Figure 5: Detected edges with 3×3 FIR filter, $\sigma^2 = \frac{1}{10}$, $r = 2$

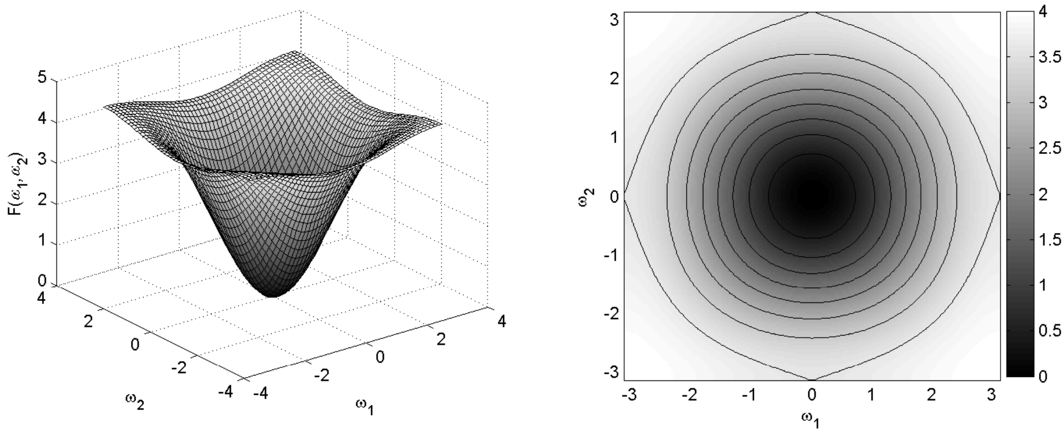

Figure 6: Spectrum of 3×3 FIR filter, $\sigma^2 = \frac{2}{9}$, $r = 2$

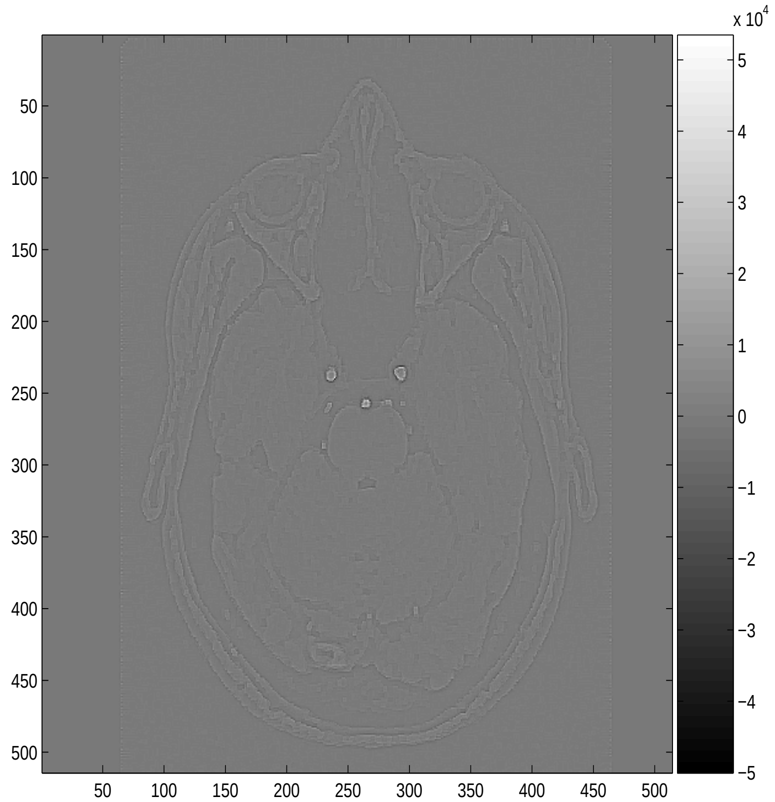

Figure 7: Detected edges with 3×3 FIR filter, $\sigma^2 = \frac{1}{10}$, $r = 2$

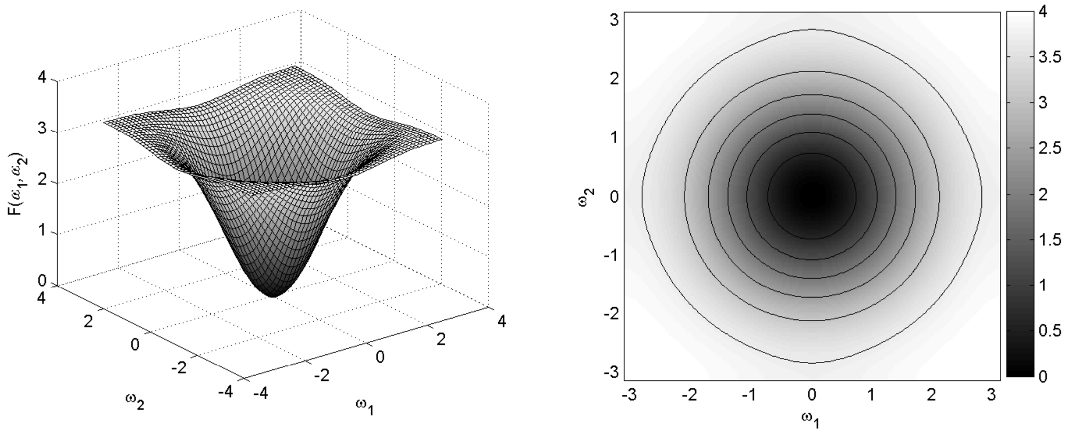

Figure 8: Spectrum of 5×5 FIR filter, $\sigma^2 = \frac{2}{9}$, $r = 2$

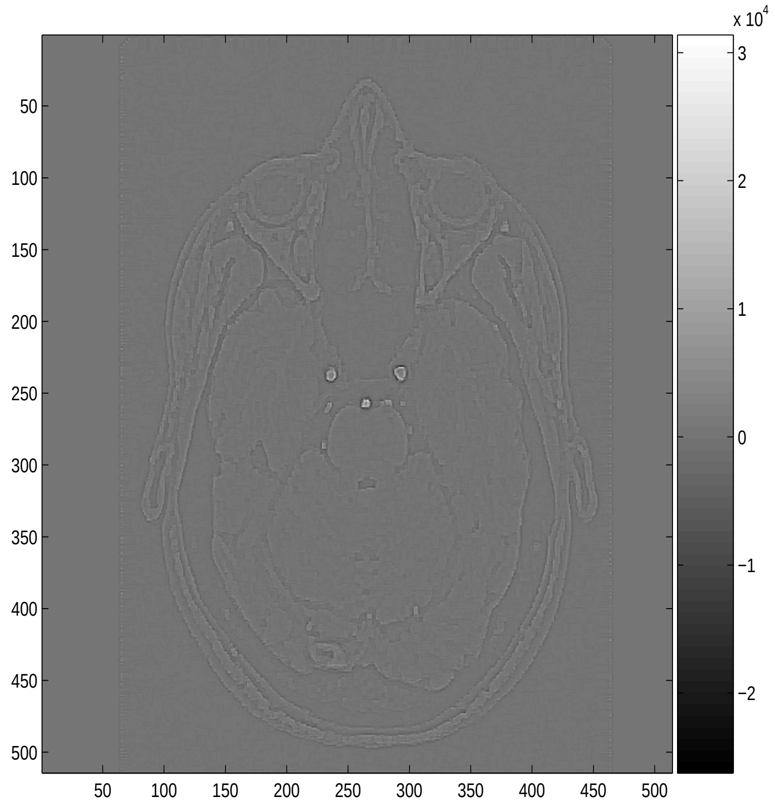

Figure 9: Detected edges with 5×5 FIR filter, $\sigma^2 = \frac{2}{9}$, $r = 2$

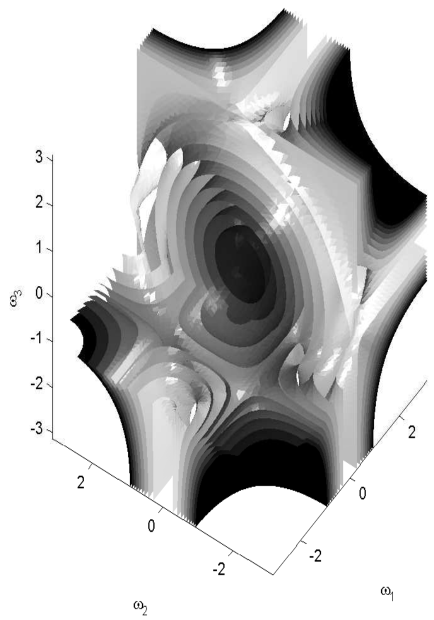

Figure 10: Spectrum of $3 \times 3 \times 3$ FIR filter, $\sigma^2 = \frac{2}{9}$, $r = 2$

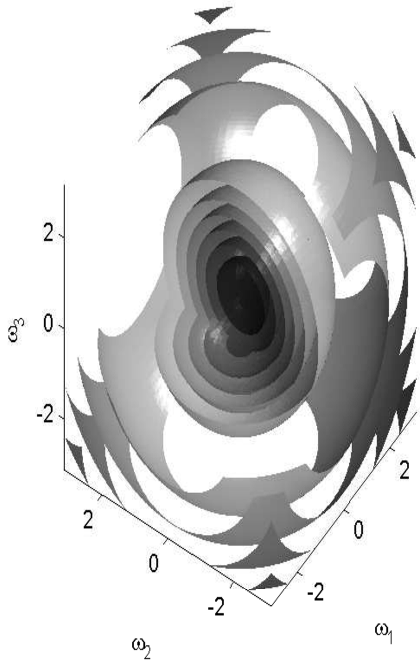

Figure 11: Spectrum of $3 \times 3 \times 3$ LoG filter, $\sigma^2 = \frac{2}{9}$, $r = 2$

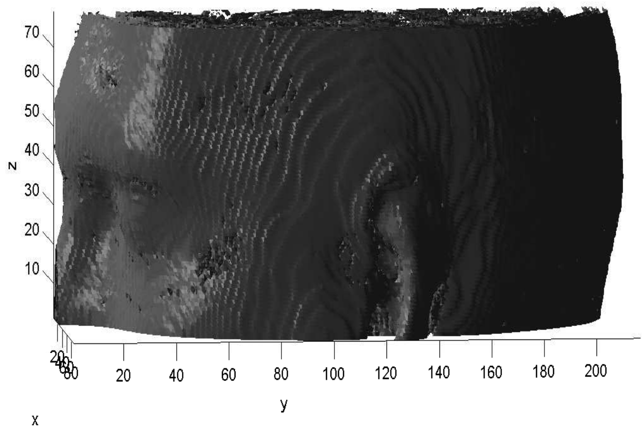

Figure 12: Detected edges with $3 \times 3 \times 3$ FIR filter, $\sigma^2 = \frac{2}{9}$, $r = 2$

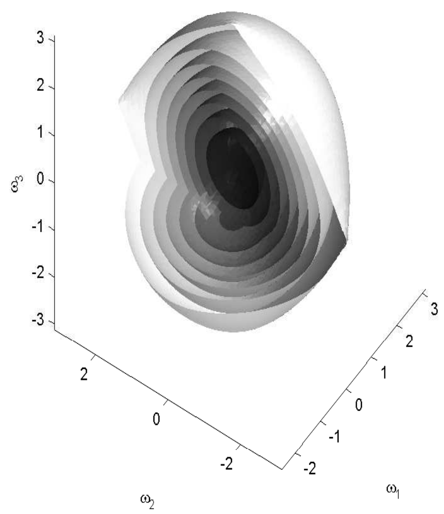

Figure 13: Spectrum of $3 \times 3 \times 3$ FIR filter, $\sigma^2 = \frac{1}{6}$, $r = 2$

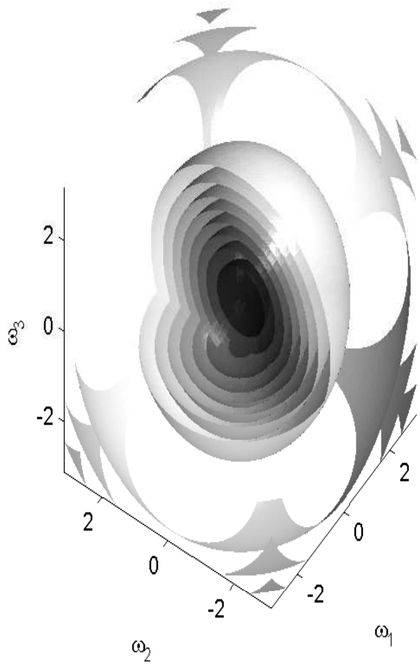

Figure 14: Spectrum of $3 \times 3 \times 3$ LoG filter, $\sigma^2 = \frac{1}{6}$, $r = 2$

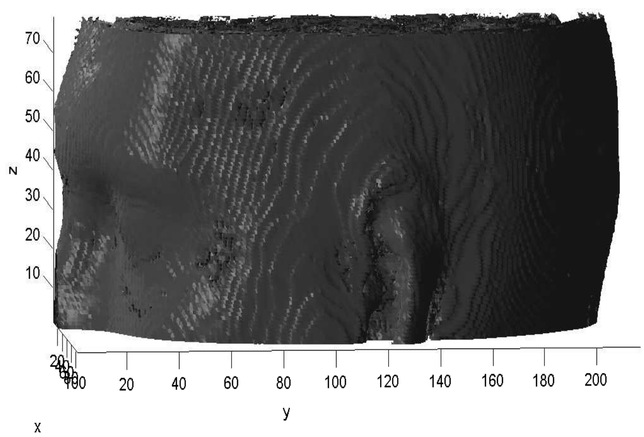

Figure 15: Detected edges with $3 \times 3 \times 3$ FIR filter, $\sigma^2 = \frac{1}{6}$, $r = 2$