

ON THE INVERSE RADON TRANSFORM

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Abstract

A reconstruction of an image by application of the discrete inverse Radon transform realized by the function `iradon` of MATLAB may be unrealistic if the input data are represented by a small number of the the image projections. To enhance the result, the set of input projections is extended, including additional virtual projections obtained from the former by a trigonometrical interpolation.

1 Introduction

The inverse Radon transform reconstructs an image from its projections along various directions. A discrete variant of the inverse Radon transform is realized by function `iradon` involved in Image Processing Toolbox of MATLAB. The paper is based upon experience with the usage of `iradon` in connection with diagnostics of plasma radiation [1].

The intent is to determine the distribution of radiation intensity in a plasma jet. It cannot be measured directly but it can be reconstructed from orthogonal projections of the radiation into planes outside the plasma jet and parallel to the plasma torch axis and placed at the same distance from it. (Mathematically, the integral of radiation intensity along a perpendicular to a projection plane is the side-on intensity in the projection plane at the foot of the perpendicular.) Introduce cylindrical coordinates (r, θ, z) such that the z -axis coincides with the torch axis. There exists $R > 0$ such that the radiation intensity vanishes if $r \geq R$. Denote $p(\theta)$ the projection plane such that the angle of the direction of its normals is θ . To describe the projections, Cartesian coordinates (x, z') are introduced in every plane $p(\theta)$ such that z' is the projection of z -axis into $p(\theta)$. Fixing a value of z -coordinate, we restrict ourselves to a specified plasma layer perpendicular to the torch axis.

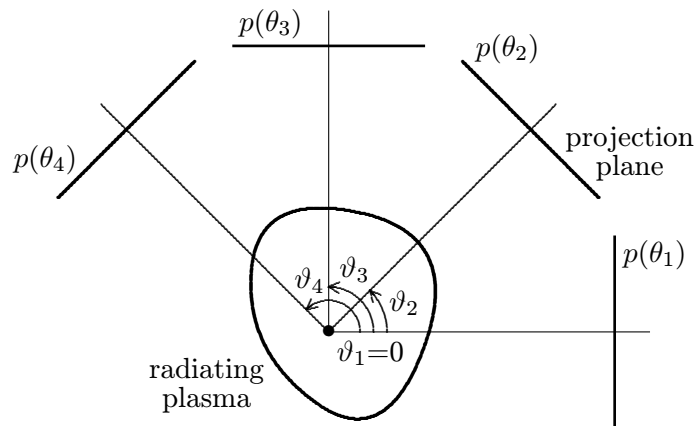


Figure 1: Scheme of detecting side-on intensity profiles of a layer of radiating plasma

To record a side-on profile of the radiation intensity, the light emitted by the plasma is projected by lens onto a detector—e.g. CCD sensor in a projection plane; see Figure 1. If there is a sufficient number of the side-on profiles at our disposal, we can apply a discrete inverse Radon transform to reconstruct mathematically the distribution of radiation intensity in the layer of the plasma jet. For this purpose, the function `iradon`, involved in MATLAB, can be applied. The more side-on profiles, the more accurate result; a serious difficulty may arise if the number of detected side-on profiles is very small.

2 Method

Consider an increasing succession of $K + 1$ angles θ_k equidistantly spread on the interval $[0, \pi]$ such that the angles θ_1 and θ_{K+1} coincide with the endpoints of this interval, and construct K projection planes $p(\theta_k)$, $k = 1, 2, \dots, K$. Given a positive integer N , let $I(\cdot; \theta_k)$ denote an N -th degree polynomial on $[-R, R]$ that approximates the side-on profile detected in the projection plane $p(\theta_k)$ so that

$$I(x; \theta_k) = \sum_{i=0}^N c_i(\theta_k) x^i \quad \text{for } \theta_k = \frac{\pi(k-1)}{K}, \quad -R \leq x \leq R, \quad k = 1, 2, \dots, K. \quad (1)$$

To determine the coefficients $c_i(\theta_k)$, the function `polyfit` involved in MATLAB can be applied. Note that `polyfit` uses the least squares method. Attention should be paid to the choice of the degree N of the approximating polynomial:

- if it is too small, the approximation does not correspond to the shape of the detected side-on profile properly,
- if it is too large, the graph of the approximation tracks the measured data points so that effect of possible errors of the measurements at individual data points is included.

Even if the polynomial approximation is satisfactory enough at the data points, there may appear unrealistic oscillations between them. Such an effect can be frequently observed near the ends of side-on profiles. To avoid such a behaviour, between the data points some interpolated values may be added. Figures 2–4 demonstrate polynomial approximations of a certain side-on profile for various N , using only measured data points (a) or using refined data points consisting of both measured and interpolated values (b, c). The interpolated values are not drawn.

Let us extend the set of angles θ_k to be defined also for $k = K + 1, K + 2, \dots, 2K$, setting $\theta_{K+k} = \theta_k + \pi$, $k = 1, 2, \dots, K$. In addition to the projection planes $p(\theta_k)$, $k = 1, 2, \dots, K$ let us consider also “virtual” projection planes $p(\theta_k)$, $k = K + 1, K + 2, \dots, 2K$. The side-on profiles in planes $p(\theta_k)$ and $p(\theta_k + \pi)$ are mutually flipped; in accordance with this fact we define $I(x, \theta_k) = I(-x, \theta_{K+k})$, $-R \leq x \leq R$, $k = 1, 2, \dots, K$. Therefore,

$$\sum_{i=0}^N c_i(\theta_k) x^i = \sum_{i=0}^N c_i(\theta_{K+k}) (-1)^i x^i, \quad -R \leq x \leq R, \quad k = 1, 2, \dots, K. \quad (2)$$

An approximation of the side-on profile for an *arbitrary* angle θ will be sought in the form

$$I(x; \theta) = \sum_{i=0}^N c_i(\theta) x^i, \quad -R \leq x \leq R \quad (3)$$

where c_i are smooth functions; they are required 2π -periodic since projection planes do not change their position when rotated by an integer multiple of 2π around the torch axis. Furthermore, the values of c_i at the points θ_k must be equal to the values $c_i(\theta_k)$, respectively, for each $k = 1, 2, \dots, 2K$. Therefore, it is natural to define the function $c_i(\theta)$ on the interval $[0, 2\pi)$ as a trigonometrical interpolation of the data $c_i(\theta_k)$, $k = 1, 2, \dots, 2K$. Following [2]

$$c_i(\theta) = \frac{1}{2} a_{i0} + \sum_{m=1}^{K-1} (a_{im} \cos m\theta + b_{im} \sin m\theta) + \frac{1}{2} a_{iK} \cos K\theta, \quad i = 0, 1, \dots, N, \quad 0 \leq \theta < 2\pi \quad (4)$$

where

$$a_{im} = \frac{1}{K} \sum_{k=1}^{2K} c_i(\theta_k) \cos m\theta_k, \quad i = 0, 1, \dots, N, \quad m = 0, 1, \dots, K$$

and

$$b_{im} = \frac{1}{K} \sum_{k=1}^{2K} c_i(\theta_k) \sin m\theta_k, \quad i = 0, 1, \dots, N, \quad m = 1, 2, \dots, K - 1.$$

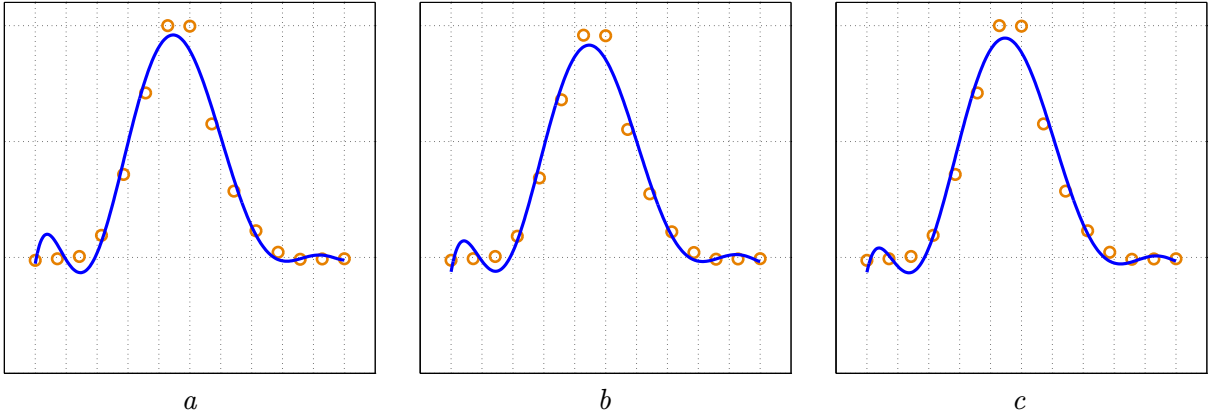


Figure 2: Measured data points of a side-on profile (circles) and relevant polynomial approximation (lines) for $N = 7$; a —only measured data points, b —measured + interpolated data points (1 interpolated value between each pair of neighbouring measured values), c —measured + interpolated data points (3 interpolated values between 1st and 2nd measured data points, 1 interpolated value between 2nd and 3rd measured data points, and similarly at the other end of profile)

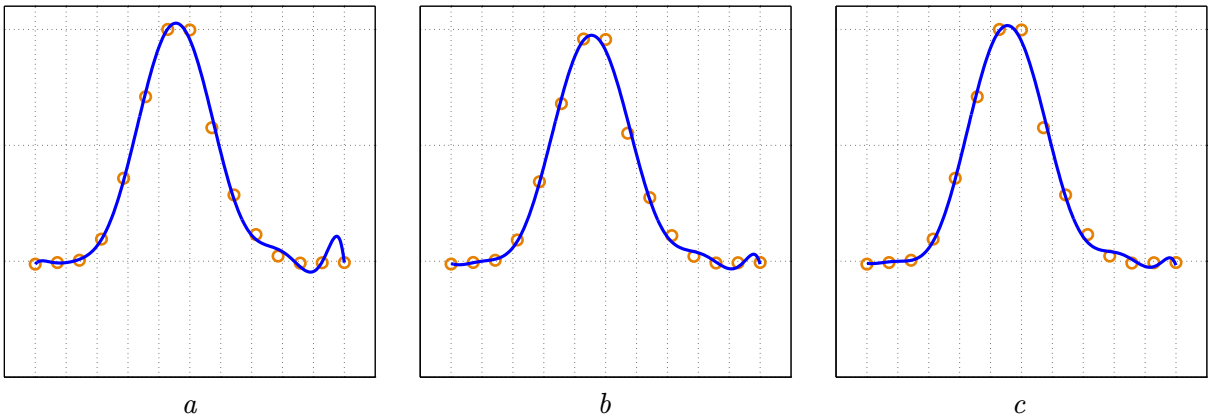


Figure 3: Measured data points of a side-on profile (circles) and relevant polynomial approximation (lines) for $N = 10$; for a , b , and c see the legend in caption to Figure 2

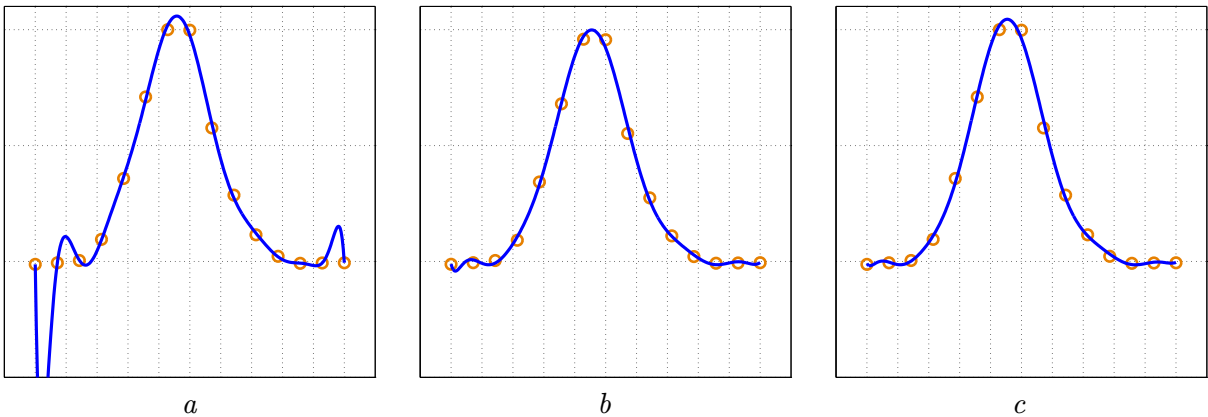


Figure 4: Measured data points of a side-on profile (circles) and relevant polynomial approximation (lines) for $N = 13$; for a , b , and c see the legend in caption to Figure 2

Considering the relation $c_i(\theta_{K+k}) = (-1)^i c_i(\theta_k)$, $k = 1, 2, \dots, K$, which follows from (2), and the relations $\cos m\theta = (-1)^m \cos m(\pi + \theta)$, $\sin m\theta = (-1)^m \sin m(\pi + \theta)$ that hold for every integer m , we can rewrite the formulas for a_{im} and b_{im} in the form

$$a_{im} = \frac{1 + (-1)^{i+m}}{K} \sum_{k=1}^K c_i(\theta_k) \cos m\theta_k, \quad i = 0, 1, \dots, N, \quad m = 0, 1, \dots, K$$

and

$$b_{im} = \frac{1 + (-1)^{i+m}}{K} \sum_{k=1}^K c_i(\theta_k) \sin m\theta_k, \quad i = 0, 1, \dots, N, \quad m = 1, 2, \dots, K-1,$$

or, using a more vivid notation,

$$a_{im} = \begin{cases} \frac{2}{K} \sum_{k=1}^K c_i(\theta_k) \cos m\theta_k & \text{if } i+m \text{ is even,} \\ 0 & \text{if } i+m \text{ is odd,} \end{cases} \quad i = 0, 1, \dots, N, \quad m = 0, 1, \dots, K \quad (5)$$

and

$$b_{im} = \begin{cases} \frac{2}{K} \sum_{k=1}^K c_i(\theta_k) \sin m\theta_k & \text{if } i+m \text{ is even,} \\ 0 & \text{if } i+m \text{ is odd,} \end{cases} \quad i = 0, 1, \dots, N, \quad m = 1, 2, \dots, K-1. \quad (6)$$

The formula (3) represents approximations for side-on profiles in directions related to angles $0 \leq \theta < 2\pi$; the appropriate coefficients $c_i(\theta)$ are given by (4) with (5) and (6). Practical usage of these formulas consists in the following. Picking a positive integer M , the system of approximate side-on profiles is extended to

$$I(x; \theta_l) = \sum_{i=0}^N c_i(\theta_l) x^i \quad \text{for } \theta_l = \frac{\pi(l-1)}{MK}, \quad -R \leq x \leq R, \quad l = 1, 2, \dots, MK. \quad (7)$$

Note that the system (1) of approximations to the measured profiles is thus included in the refined system (7).

Practically, we apply the procedure `iradon` successively with $M = 2^j$ discretized side-on profiles where $j = 0, 1, 2, \dots$. (Note that for $j = 0$, just the K input side-on profiles arising from measurement are used.) Increasing j by 1, the number of input profiles is doubled. As soon as the difference between the reconstructed radiation intensity in the last two steps is negligible, the process can be finished.

3 Example

Consider a model radiation intensity whose isolines are presented on Figure 5a. Using the discrete (forward) Radon transform realized by the function `radon` contained in MATLAB environment, side-on profiles simulating measured values are obtained. The first of them, corresponding to the angle $\theta = 0$, is represented by circles on Figures 5b and 5c. Polynomial approximations are constructed for $N = 10$; on the former picture using only the considered data points, on the latter also interpolated values between them. Only such enhanced approximations of profiles are employed in the sequel. Reconstructions of the model radiation intensity are performed following the presented method. On Figure 6 we start with two simulated profiles (a), then we add two (b) and six (c) interpolated profiles. Evidently, the reconstruction on the first picture is quite useless, while those on the other pictures are improved considerably. Now we repeat the process starting with more simulated profiles: Figure 7 shows the results for four simulated profiles (a) and also for these profiles with four (b) and twelve (c) additional interpolated ones. The

reconstruction on the first picture is still unsatisfactory (even though better than on Figure 6a), but the reconstructions using the interpolated profiles now match the original model.

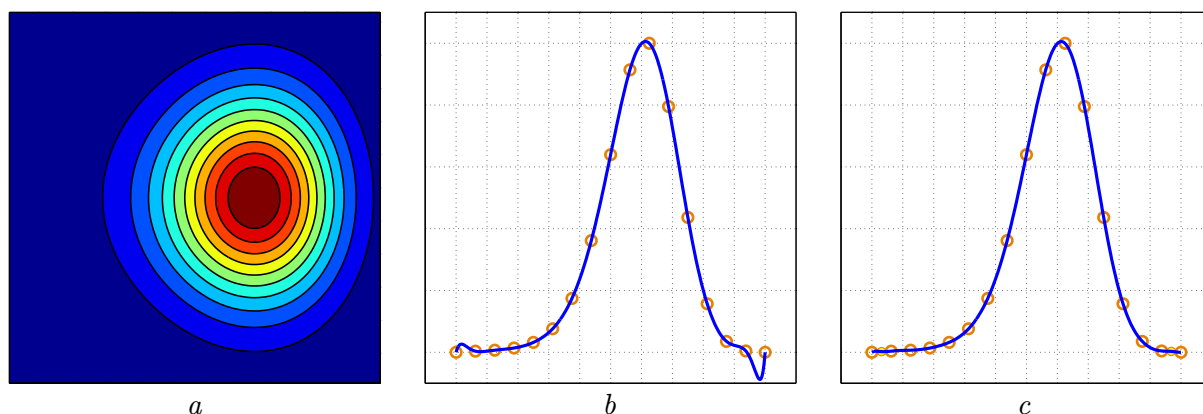


Figure 5: Model radiation intensity; *a*—isolines, *b*—polynomial approximation (line) of side-on profile for $\theta_1 = 0$ using only data points obtained by forward Radon transform (circles), *c*—enhanced polynomial approximation (line) of the same profile using the former data points and interpolated values

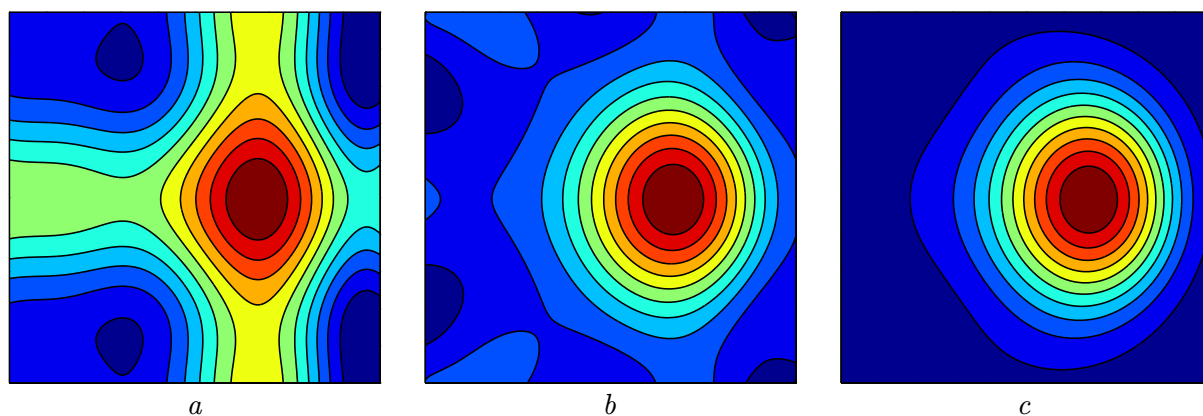


Figure 6: Reconstruction of radiation intensity by `iradon`; *a*—2 simulated side-on profiles, *b*—2 simulated + 2 interpolated side-on profiles, *c*—2 simulated + 6 interpolated side-on profiles

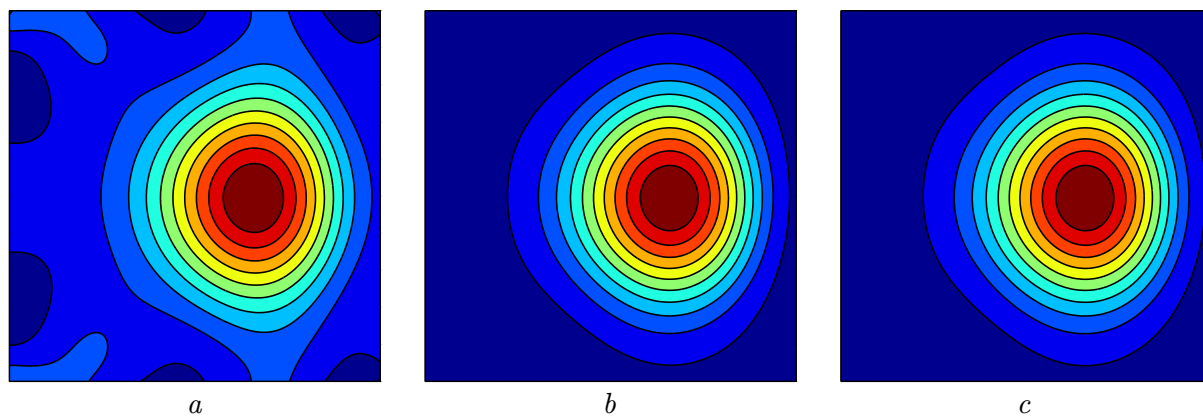


Figure 7: Reconstruction of radiation intensity by `iradon`; *a*—4 simulated side-on profiles, *b*—4 simulated + 4 interpolated side-on profiles, *c*—4 simulated + 12 interpolated side-on profiles

4 Conclusion

An application of the inverse Radon transform to projections of an image realizes the image reconstruction. A discrete variant of the algorithm can be carried out by the function `iradon` involved in the programming environment MATLAB. However, the result may be unrealistic if the number of the projections is small. The paper concerns a possibility of improving the result. The method consists in increasing the number of input side-on profiles by constructing additional fictive profiles corresponding to other directions of projecting; the recorded data of measured profiles are approximated by polynomials (of the same degree) and then a trigonometrical interpolation of their coefficients leads to polynomial approximations of the fictive projections. Although the interpolated profiles do not add a new information on the recorded side-on data, the use of them may enhance the result of reconstruction, which was demonstrated by a model example.

Acknowledgement

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References

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- [2] A. Ralston. *A first course in numerical analysis*. McGraw-Hill, New York, 1961

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