PROJECTILE OPTIMIZATION VIA NEWTON'S MODEL IN MATLAB

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Abstract

When the radial symmetric projectile works through a very sparse gas of constant density with fixed velocity, the friction force can be approximated via Newton's model. This is based on the assumption of the gas flow as independent movement of non–interacting mass particles, which hit the projectile shape and change their momentum. The absence of particle interactions is in the contradiction to laminar and turbulent flow models, which are preferred for dense fluids. Thus the original Newton model is only useful for two occasions: motion in low pressure gas and generating of good tasks for the calculus of variation. Our paper is oriented to various techniques how to obtain the best possible solution of given variation task. The role of Matlab environment was in the automation of symbolic computations (differentiation, integration), numeric integration (Simpson's rule), parameter optimization (conjugate gradient) and graphical presentation of results (geometry of optimum neighborhood, optimum projectile design).

1 Original Form of Newton's Model

When the radial symmetric projectile of radius R and length W works through a very sparse gas of density ρ having the velocity v, the friction force can be approximated via Newton's model [1] as

$$F = 4\pi\rho v^2 \int_0^W \frac{\mathrm{y}(x)\,\mathrm{y}_\mathrm{x}^3(x)\,\mathrm{d}x}{1+\mathrm{y}_\mathrm{x}^2(x)}$$

Here y: $[0,W] \rightarrow [0,R]$ is unknown non–decreasing (projectile profile) function satisfying $\mathrm{y}(0) = 0$, $\mathrm{y}(W) = R$ and y_x is its differentiation. This a little bit naive model is based on the assumption of the gas flow as independent movement of non–interacting mass particles, which hit the projectile shape and change their momentum. The absence of particle interactions is in the contradiction to laminar and turbulent flow models, which are preferred for dense fluids. Thus the original Newton model is only useful for two occasions: motion in low pressure gas and generating of good tasks for the calculus of variation.

2 Various Approaches to Variational Task

There are many additional assumptions, which enable to solve the basic variation task [1]

$$\mathrm{I}_1(\mathrm{y}) = \int_0^W \frac{\mathrm{y}(x)\,\mathrm{y}_\mathrm{x}^3(x)\,\mathrm{d}x}{1+\mathrm{y}_\mathrm{x}^2(x)} = \min \quad \text{subject to} \quad \mathrm{y}(0)=0,\ \mathrm{y}(W)=R$$

It corresponds with the minimum friction force design of given projectile in a sparse gas. The first assumption of the continuity and smoothness of $\mathrm{y}(x)$, is useful for the traditional solution via calculus of variations. Unfortunately, the resulting Euler's differential equation [1] of the first order

$$\mathrm{y}(x) = C_1 \frac{\mathrm{y}_\mathrm{x}(x)+1}{\mathrm{y}_\mathrm{x}^3(x)\left(\mathrm{y}_\mathrm{x}^2(x)+1\right)} > 0$$

has no exact (analytical) solution and the positive value of y(x) is in the contradiction to the condition y(0) = 0. Thus, the assumption of continuity and smoothness is probably the main problem of the original task.

The additive condition $y_x(x) << 1$ is a "little bit false" tool for analytical solution of alternative variational task [1]

$$I_2(y) = \int_0^W y(x)\,y_x^3(x)\,dx = \min \quad \text{subject to} \quad y(0) = 0,\ y(W) = R$$

The adequate Euler's equation has analytical solution $y^*(x) = R(x/W)^{3/4}$, but with $y_x(x) >> 1$ for small x, which is rather illustrative or referential then exact solution of the original task.

Our paper is oriented to various techniques how to obtain the best possible approximate solution of given variational task. But we prefer the reformulation of the original functional $I_1(y)$ to various but equivalent forms, which can have different properties in the process of numeric integration.

Our first model includes the possible discontinuity of the projectile shape y(x) in the point x = 0. It corresponds to the projectile with planar circular toe of radius $r < R$. The original functional $I_1(y)$ can be decomposed to a sum of two integrals. Using the Symbolic toolbox in the Matlab environment, we simplify the first integral to $r^2/2$ an then reformulate $I_1(y)$ to the new form

$$I_3(y) = \frac{r^2}{2} + \int_0^W \frac{y(x)\,y_x^3(x)\,dx}{1 + y_x^2(x)} = \min \quad \text{subject to} \quad y(0) = r,\ y(W) = R$$

Here, the function y(x) is continuous and smooth everywhere and the radius r of "dummy toe" is also subject of optimization.

When we would like to prevent the problem of discontinuity in the coordinate origin (0,0) otherwise, we can invert the function y(x) to x(y). After a few symbolic computations with $I_1(y)$, we obtained the other but equivalent variational task

$$I_4(x) = \int_0^R \frac{y\,dy}{1 + x_y^2(y)} = \min \quad \text{subject to} \quad x(0) = 0,\ x(R) = W$$

which has a solution x(y) as non–decreasing continuous function with constrained differentiation $x_y(y)$. The adequate Euler's differential equation of the first order has the form

$$y = C_2 \frac{\left(x_y^2(y) + 1\right)^2}{x_y(y)} > 0$$

The analytical solution is impossible. But we investigated that the optimum solution has a single breakpoint at $y = r$, where the differentiation does not exist. The optimum solution has two parts: constant function x(y) = 0 for $y \le r$ and smooth increasing function x(y) for $y \ge r$.

To avoid the numerical problems in the numeric integration of non–smooth function x(y), we also decompose $I_4(x)$ to the sum of two integrals, where the first one is trivial. The last form of variation task is expressed as

$$I_5(x) = \frac{r^2}{2} + \int_r^R \frac{y\,dy}{1 + x_y^2(y)} = \min \quad \text{subject to} \quad x(r) = 0, x(R) = W$$

From the numeric point of view, only the functionals $I_3(y)$ and $I_5(x)$ are the improved equivalents of $I_1(y)$. The functional $I_2(y)$ is not equivalent form but only weak approximation of $I_1(y)$. The numeric integration of $I_4(x)$ via fixed step Simpson's method has unacceptable error order.

3 Approximation of Projectile Shape

The absence of analytical solution is evident for the optimum projectile task. But we can approximate the projectile shape via various models and then compare the quality of approximations. The first model is only a generalization of the function $y^*(x)$ with single free parameter $\alpha > 0$ in the form $y_A(x) = R(x/W)^{\alpha}$ which is useful for the minimization of functional $I_1(y)$. The second model is simple linear approximation $y_B(x) = r + (R - r)x/W$ for the functional $I_3(y)$. This model has also a single parameter r. The third model $x_C(y) = W(y - r)/(R - r)$ is also the linear model but for the functional $I_5(y)$. These three trivial models are useful for the symbolic computation. The value of $I_1(y_A)$ can be calculated exactly (analytically) for many rational values of parameter $\alpha = m/n$. The values of $I_3(y_B)$ and $I_5(x_C)$ were also analytically calculated in Matlab Symbolic Toolbox as

$$I_3(y_B) = I_5(x_C) = \frac{r^2}{2} + \frac{1}{2}\frac{(R-r)^3(R+r)}{(R-r)^2 + W^2}$$

These three models can generate simple tasks for the single parameter minimization (exact or numeric). The other three models were created as polynomial extension of previous three ones

$$y_D(x) = y_A(x) + \sum_{k=1}^{n} a_k x^k (W - x)$$

$$y_E(x) = y_B(x) + \sum_{k=1}^{n} b_k x^k (W - x)$$

$$x_F(y) = x_C(y) + \sum_{k=1}^{n} c_k (y - r)^k (R - y)$$

Now we are prepared to follow in the research direction of study [2] and try to find the better formulas for the optimum projectile profile.

4 Numeric Experiments

All the numeric experiments were performed numerically in the Matlab environment with extended models y_D, y_E, x_F and functionals I_1, I_3, I_5 using analytical differentiation but numeric integration by Simpson's method with fixed step for $N = 2000$. It is not too correct for the functional I_1 due to possible numeric difficulties near $x = 0$. The models y_A, y_B, x_C are only special cases of previous ones for $n = 0$. The functionals were numerically minimized via `fmincon` function from Optimization Toolbox for given models and various orders. The numeric results for $R = W = 1$ are collected in the *Tab. 1*. It is useful to compare the results with trivial analytical values for special cases of projectile shape. The worst case of cylindrical profile corresponds to linear model y_B with $r = 1$ and has value $I_3(y_B) = ½ = 0.500000000$. The conic toe shape corresponds to linear model y_B with $r = 0$ and has value $I_3(y_B) = ¼ = 0.250000000$, which is significant improvement (that is why the projectile toe is useful). The hemispheric shape has the same value of functional I_1 as the conic one. The profile y^* (*Fig. 1*) which is recommended in [1] is represented by model y_A with $\alpha = ¾$ and has the value $I_1(y_A) = 0.220013580$. The model y_A with $\alpha = ½$ represents parabolic shape with the better analytic value $I_1(y_A) = 0.201013072$ and should be easily realized in practice. As seen in the *Tab. 1*, the best model of the class y_A has $\alpha = 0.462426587$ and the value $I_1(y_A) = 0.200705898$ with the shape demonstrated on the *Fig. 2*. The best linear model of the class y_B has $r = 0.381966029$ and the value $I_3(y_B) = 0.200705898$ with the shape demonstrated on the *Fig. 3*. The numerical experiments in [2] corresponds to the model y_E with $r = 0$ and has the functional values 0.250000, 0.223355, 0.213719, 0.208475 for n =0, 1, 2, 3. The new results collected in the *Tab. 1* are better then previous ones including trivial cases. The final optimum shapes for $n = 3$ are depicted on the *Figs. 4 – 6*.

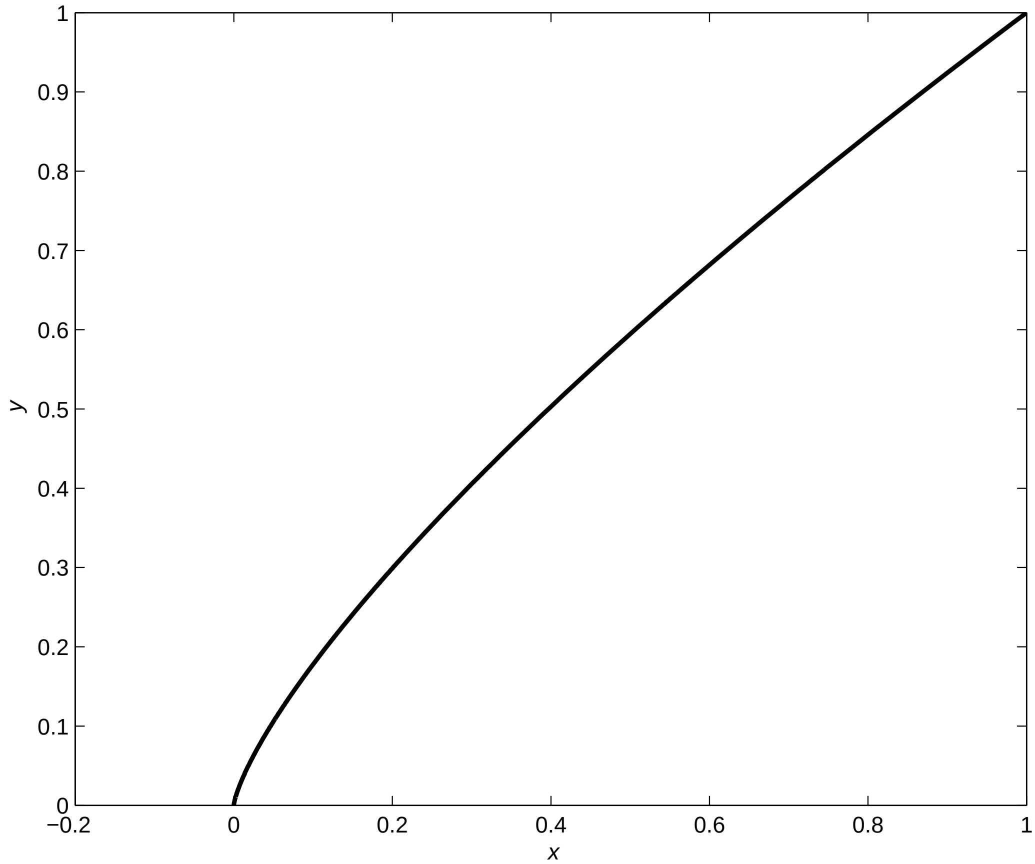

Figure 1: Optimum projectile y*

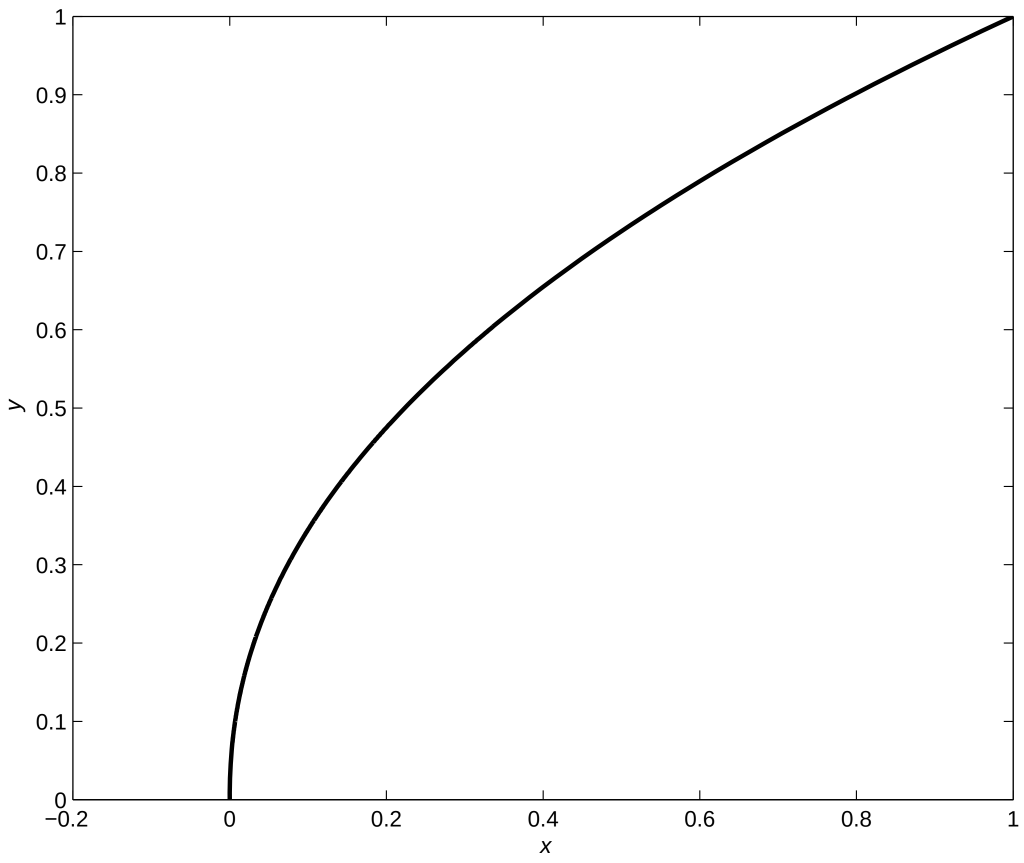

Figure 2: Optimum projectile y_A

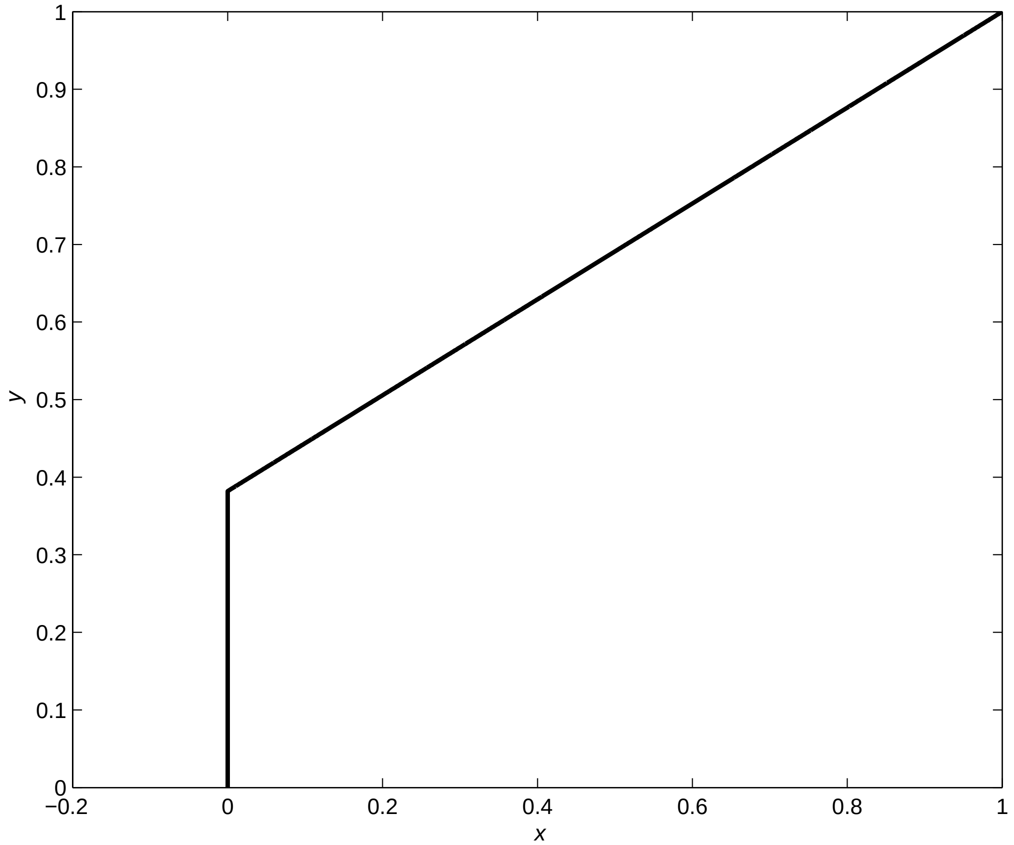

Figure 3: Optimum projectile y_B or x_C respectively

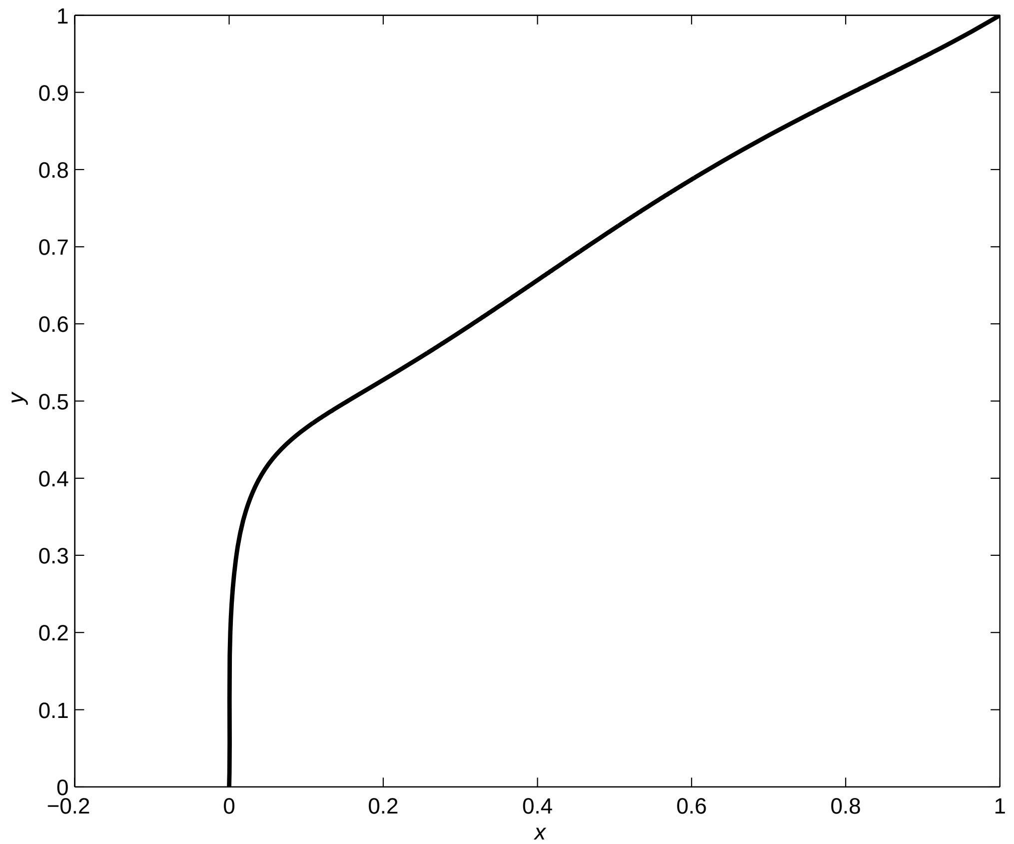

Figure 4: Optimum projectile y_D

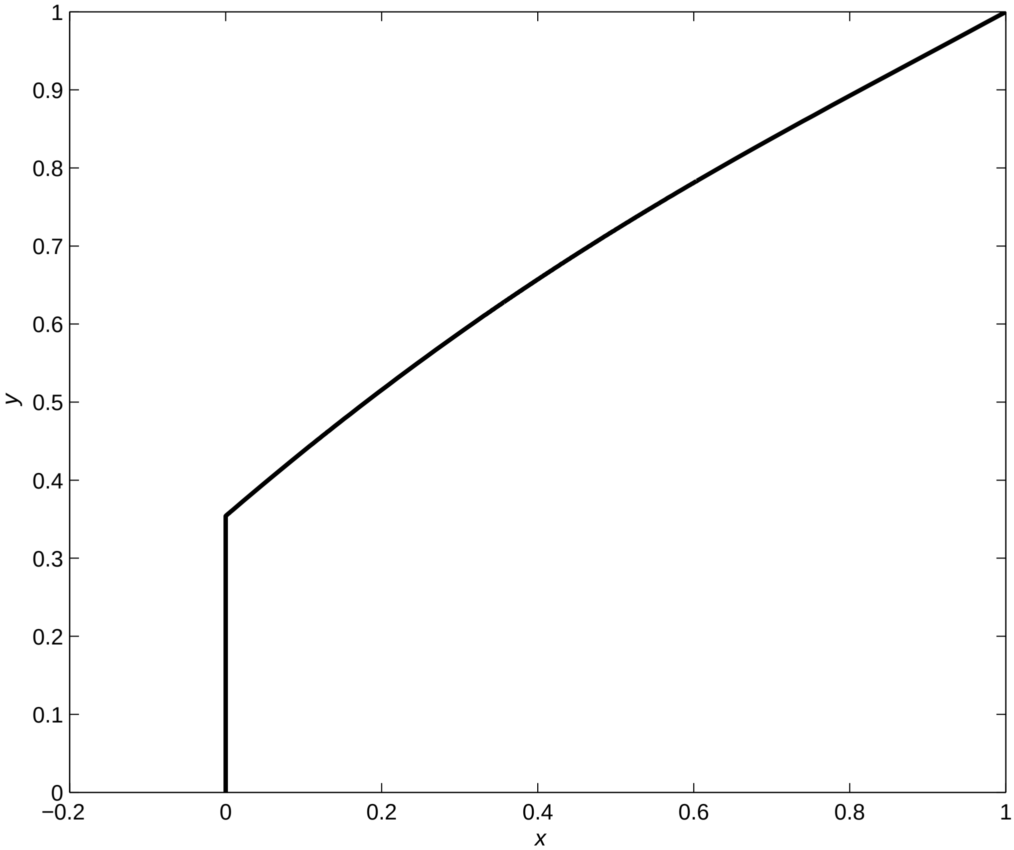

Figure 5: Optimum projectile y_E (the best included)

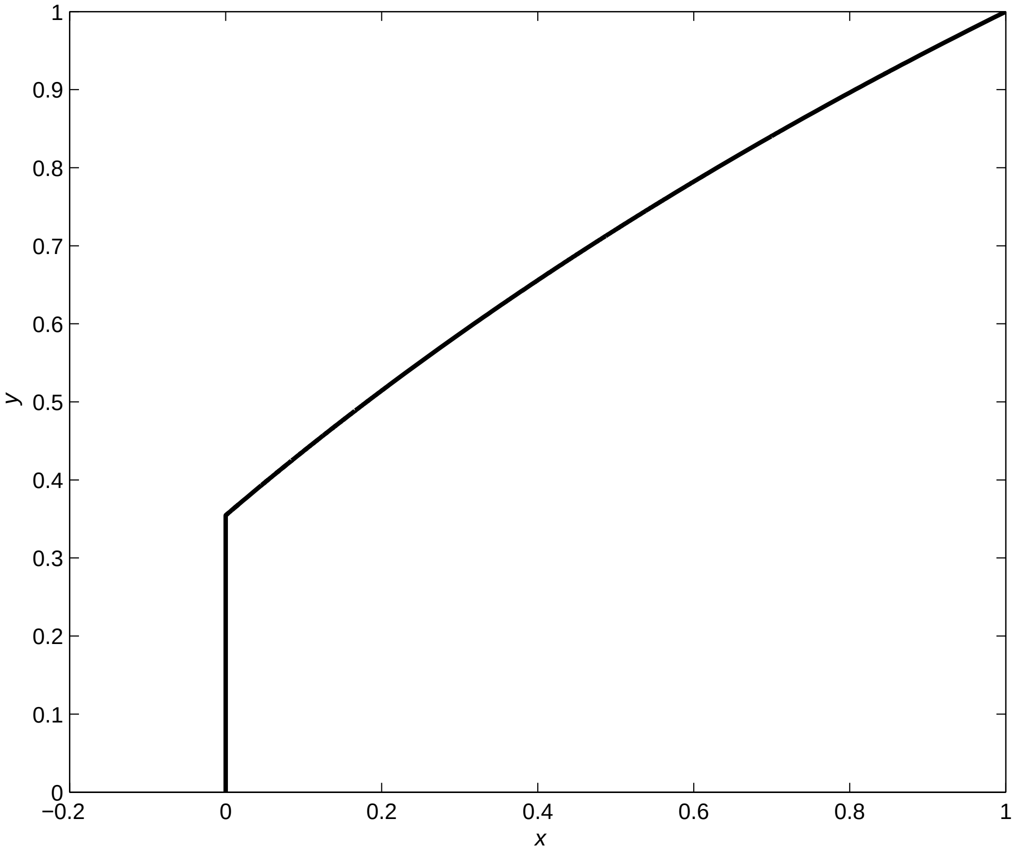

Figure 6: Optimum projectile x_F

Table 1: QUALITY OF OPTIMUM PROJECTILE SHAPES

n	$I_1(y_D)$ (α_{opt})	$I_3(y_E)$ (r_{opt})	$I_5(x_F)$ (r_{opt})
0	0.200705898 (0.462426587)	0.190983006 (0.381966029)	0.190983006 (0.381966029)
1	0.195440564 (0.342541303)	0.187804668 (0.358007030)	0.187593024 (0.354594021)
2	numeric difficulties	0.187540245 (0.354064327)	0.187570458 (0.354301827)
3	numeric difficulties	0.187473556 (0.352718593)	0.187481221 (0.352443117)

5 Results

The optimum projectile shape was studied first by Isaac Newton in 1687. This task is a good inspiration for effective approximation at present time. We reformulate the original task to avoid numerical difficulties during calculations and then we tested various models of projectile shape. Our paper is about discontinuity analysis, modeling and parameter optimization. The best included solution for $R = W = 1$ and $n = 3$ is the model y_E with functional value $I_3(y_E) = 0.187473556$ and circular planar toe of radius $r = 0.352718593$. The role of Matlab environment was in the automation of symbolic computations, numeric integration, parameter optimization and graphical presentation of results.

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Reference

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