RATIONAL APPROXIMATION OF TIME DELAY

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Abstract

Real dynamical systems often show some time lag between a change of an input and the corresponding change of the output. This time lag has a whole range of causes. For needs of mathematical modelling, it is aggregated into a total phenomenon called time delay or dead time. If a real system with time delay is modelled as a time invariant linear system, its transfer function (rational function) becomes due to time delay a transcendental function. Most of methods used for analysis and synthesis of control systems are developed for transfer functions in the form of rational functions only. If these methods have to be used also for dynamical systems with time delay then it is necessary to approximate transfer functions of time delay by means of rational functions. Approximations usually use either Padé approximation or Taylor series of the exponential function. A simple modification of Padé approximation application is given in the contribution. It improves its time behaviour not only close to the origin but also in longer time ranges.

1 Introduction

A time lag between a change of an input and the corresponding change of the output that real dynamical systems often show has a whole range of causes. For needs of mathematical modelling, it is aggregated into a total phenomenon called time delay or dead time. Time delay can be defined as a delay attributable to the time taken in the transporting material or to the finite rate of propagation of a signal. Dead time is a time interval between the instant when the variation of an input variable is produced and the instant when the consequent variation of the output variable starts. Time delays are sometime structured by places of their genesis: input time delay, dead time of the system and output time delay. This classification is significant only when using state-space description of systems. If transfer functions are used then time delays cannot be structured in such a way.

If a dynamical system with time delay is modelled as a time invariant linear system, its transfer function (rational function) becomes due to time delay a transcendental function. MATLAB enables to model these systems by means of transfer function with time delay in the form:

$$G(s)=\frac{M(s)}{N(s)}=\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}\, e^{-Ts} \tag{1}$$

Most of methods used for analysis and synthesis of control systems are developed for transfer function in the form of the rational function only (e.g. algebraic methods of determining stability, root locus method, etc.). The **`rlocus`** function for example responds to an attempt to calculate and display the root locus for a system with time delay this way: **`Not supported for models with delays. Use PADE to approximate delays`**.

2 Approximation of time delay transfer functions

If the algebraic methods have to be used also for dynamical systems with time delay then the transfer function of time delay must be approximated using a suitable rational function:

$$R_{m,n}(Ts)=\frac{1-\dfrac{(\alpha sT)}{1!}+\dfrac{(\alpha sT)^2}{2!}+\cdots+(-1)^n\dfrac{(\alpha sT)^m}{m!}}{1+\dfrac{(\beta sT)}{1!}+\dfrac{(\beta sT)^2}{2!}+\cdots+\dfrac{(\beta sT)^n}{n!}} \tag{2}$$

where α, β are suitably chosen coefficients. Approximation methods usually arise from two basic approaches: either Taylor series of the exponential function or its Padé approximation.

Padé approximation $R_{n,n}(Ts)$ with polynomials of the same degree in the numerator and denominator is commonly used (function **Pade** in MATLAB).

2.1 Approximation of time delay transfer functions using Taylor series

This method of approximation of a time delay transfer function arises from development of the exponential function into Taylor (Maclaurin) series

$$e^{-sT} = 1 - \frac{(sT)}{1!} + \frac{(sT)^2}{2!} + \cdots + (-1)^n \frac{(sT)^n}{n!} + \cdots \tag{3}$$

The development by means of the Taylor series into a polynomial in the numerator $R_{n,0}(Ts)$ does not meet the conditions of physical realizability. Moreover, it introduces unstable zeroes into the transfer function. On the contrary, the development by means of the Taylor series into a polynomial in the denominator $R_{0,n}(Ts)$ meets the strong conditions of physical realizability but for higher degrees of the polynomial ($n > 4$) is unstable. There is a suitable compromise based on approximation of the time delay transfer function using a rational function. The exponential function is formally divided into two parts. The first of them is the numerator and the second one the denominator:

$$e^{-sT} = e^{-\left(\frac{T}{2}+\frac{T}{2}\right)s} = e^{-s\frac{T}{2}} e^{-s\frac{T}{2}} = \frac{e^{-s\frac{T}{2}}}{e^{s\frac{T}{2}}} \tag{4}$$

The main advantage of this modification is higher precision of the approximation also for the lower degrees of polynomials. The disadvantages are similar as for the previous methods: it meets the weak conditions of physicals realizability only, introduces unstable zeroes into the transfer function and for degrees $n > 4$ is unstable.

Table 1: TAYLOR APPROXIMATION OF TIME DELAY TRANSFER FUNCTION

Degree n	*Taylor approximation* $R_{n,n}(Ts)$
1	$\frac{2-(Ts)}{2+(Ts)}$
2	$\frac{8-4(Ts)+(Ts)^2}{8+4(Ts)+(Ts)^2}$
3	$\frac{48-24(Ts)+6(Ts)^2-(Ts)^3}{48+24(Ts)+6(Ts)^2+(Ts)^3}$
4	$\frac{384-192(Ts)+48(Ts)^2-8(Ts)^3+(Ts)^4}{384+192(Ts)+48(Ts)^2+8(Ts)^3+(Ts)^4}$
5	$\frac{3840-1920(Ts)+480(Ts)^2-80(Ts)^3+10(Ts)^4-(Ts)^5}{3840+1920(Ts)+480(Ts)^2+80(Ts)^3+10(Ts)^4+(Ts)^5}$

The Taylor approximation of the time delay transfer function has significantly different behaviour in the right neighbourhood of the origin in comparison with the time delay. While the limit of the time delay in the point $t \to 0+$ is zero, the limit of its approximation is always non-zero (Fig. 1) and depends on the degree of the approximation:

$$\lim_{t\to 0+} \overline{R}_{n,n}(t) = \lim_{p\to\infty} p\, R_{n,n}(Tp)\frac{1}{p} = \begin{cases} -1\,, n = 2k+1 \\ +1,\, n = 2k \end{cases} \tag{5}$$

This disadvantage can be removed by a relatively simple way. If the polynomial in the numerator is of lower degree than that in the denominator then the limit is always zero (Fig. 1):

$$\lim_{t\to 0+} \overline{R}_{m,n}(t) = \lim_{p\to\infty} p\, R_{m,n}(Tp)\frac{1}{p} = \begin{cases} (-1)^n\ , m = n \\ \quad 0 \quad , m < n \end{cases} \tag{6}$$

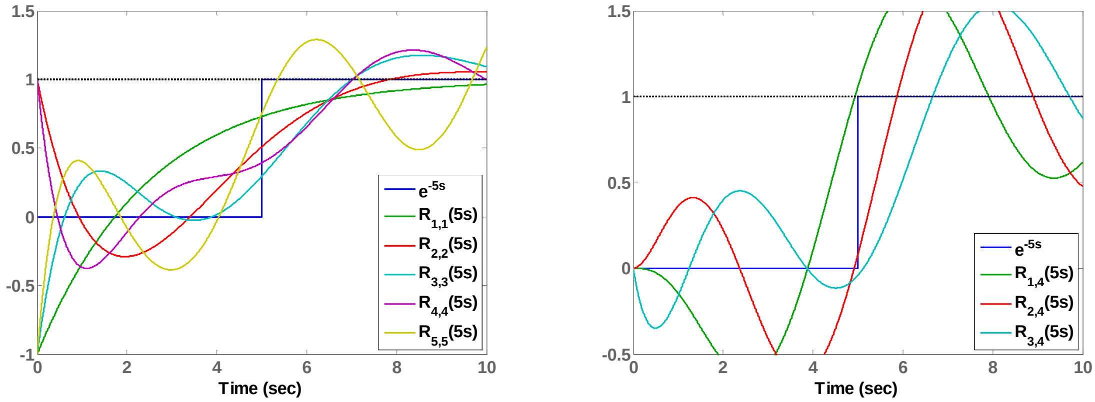

Figure 1: Taylor approximation with the same and different degrees of polynomials for the pure time delay

This improvement of the time response of Taylor approximation of the time delay transfer function in the right neighbourhood of the origin results in significant deterioration of its behaviour for higher values of time. This is because the modified rational function approximates the exponential function with a significant worse precision (see Tables 2 and 3). The integral of the square error of the approximation on the interval $< 0; 2T >$ calculated using the trapezoidal rule with the step $h = 0.001$ is used as the precision measure. All comparative calculations and graphical representations were carried out for the time delay $T = 5$ a for the third order comparative system with the unit gain and the real single poles $s_1 = -1,\ s_2 = -2,\ s_3 = -3$ (Fig. 2 and 4).

Table 2: COMPARISON OF ERRORS FOR TAYLOR APPROXIMATION OF TIME DELAY TRANSFER FUNCTION WITH SAME DEGREES OF BOTH POLYNOMIALS

Taylor approximation degrees	1/1	2/2	3/3	4/4	5/5
Pure time delay	1.3514	0.6621	0.6791	0.7919	0.9863
3rd order system with time delay	0.4444	0.081	0.1118	0.1017	0.1418

Table 3: COMPARISON OF ERRORS FOR TAYLOR APPROXIMATION OF TIME DELAY TRANSFER FUNCTION WITH DIFFERENT DEGREES OF BOTH POLYNOMIALS

Taylor approximation degrees	1/4	2/4	3/4
Pure time delay	1.9554	1.972	1.499
3rd order system with time delay	4.5712	3.2996	1.328

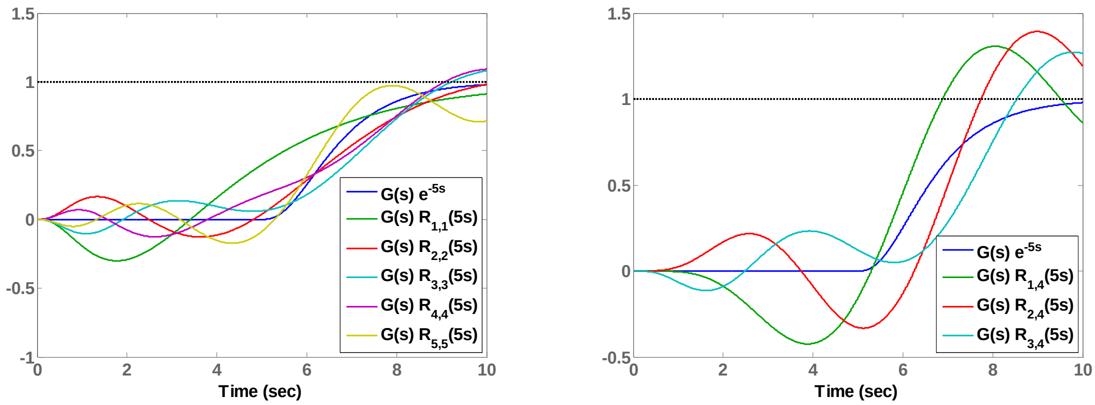

Figure 2: Taylor approximation with the same and different degrees of polynomials for the third order system with time delay

2.2 Padé approximation of time delay transfer function

Another often used method of rational approximation of time delay transfer function is application of Padé approximant. The Padé approximant enables to approximate more complex functions using rational function of the type $R_{m,n}(Ts)$. The coefficients of both polynomials are calculated so that the first $m+n+1$ terms vanish in Taylor series of the approximated function. First the exponential function is developed into the Taylor series at the origin in $m+n+1$ terms and then is approximated by a rational function ($q_0 = 0$):

$$e^{-Ts} \approx \sum_{i=0}^{m+n} (-1)^i \frac{(Ts)^i}{i!} = \frac{\sum_{i=0}^{m} p_i (Ts)^i}{\sum_{i=0}^{n} q_i (Ts)^i} \tag{7}$$

After removing the denominator a linear equation set for unknown coefficient of both polynomials can be obtained:

$$\left[\begin{array}{cccccccccccc} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -a_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -a_1 & -a_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & -a_{m-2} & -a_{m-1} & \cdots & -a_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -a_{m-1} & -a_{m-2} & \cdots & -a_1 & -a_0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & -a_m & -a_{m-1} & \cdots & -a_2 & -a_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -a_{m+1} & -a_m & \cdots & -a_3 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & -a_{m+n-2} & -a_{m+n-3} & \cdots & -a_m & -a_{m-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & -a_{m+n-1} & -a_{m+n-2} & \cdots & -a_{m+1} & -a_m \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ \cdots \\ p_{m-1} \\ p_m \\ \hline q_1 \\ q_2 \\ \cdots \\ q_{n-1} \\ q_n \end{array}\right] = \left[\begin{array}{c} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_{m-1} \\ a_m \\ \hline a_{m+1} \\ a_{m+2} \\ \cdots \\ a_{m+n-1} \\ a_{m+n} \end{array}\right] \tag{8}$$

This method has an important feature that there are no restriction on the degrees *m* and *n* of both polynomials. They may be the same but also can differ from each other. The equation set (8) can be solved in general and so the explicit formulas for the coefficients of both polynomials can be obtained. The formulas for the same degrees of polynomials ($m = n$) can be found in the form:

$$p_i = (-1)^i \frac{(2n-i)!\, n!}{(2n)!\, i!\, (n-i)!} \quad q_i = \frac{(2n-i)!\, n!}{(2n)!\, i!\, (n-i)!} \quad i = 0, 1, \ldots, n \tag{9}$$

Table 4: PADÉ APPROXIMATION OF TIME DELAY TRANSFER FUNCTION

Degree n	*Padé approximation* $R_{n,n}(Ts)$
1	$\frac{2-(Ts)}{2+(Ts)}$
2	$\frac{12-6(Ts)+(Ts)^2}{12+6(Ts)+(Ts)^2}$
3	$\frac{120-60(Ts)+12(Ts)^2-(Ts)^3}{120+60(Ts)+12(Ts)^2+(Ts)^3}$
4	$\frac{1680-840(Ts)+180(Ts)^2-20(Ts)^3+(Ts)^4}{1680+840(Ts)+180(Ts)^2+20(Ts)^3+(Ts)^4}$
5	$\frac{30240-15120(Ts)+3360(Ts)^2-420(Ts)^3+30(Ts)^4-(Ts)^5}{30240+15120(Ts)+3360(Ts)^2+420(Ts)^3+30(Ts)^4+(Ts)^5}$

Similar formulas can be gained by the same way for the different degrees ($m \neq n$) of polynomials:

$$p_i = (-1)^i \frac{(m+n-i)!\, m!}{(m+n)!\, i!\, (m-i)!}, \quad i = 0, 1, \ldots, m$$
$$q_i = \frac{(m+n-i)!\, n!}{(m+n)!\, i!\, (n-i)!}, \quad i = 0, 1, \ldots, n \qquad (10)$$

Table 5: PADÉ APPROXIMATION OF TIME DELAY TRANSFER FUNCTION

Degrees m/n	*Padé approximation* $R_{m,n}(Ts)$
1/5	$\frac{720-120(Ts)}{720+600(Ts)+240(Ts)^2+60(Ts)^3+10(Ts)^4+(Ts)^5}$
2/5	$\frac{2520-720(Ts)+60(Ts)^2}{720+600(Ts)+240(Ts)^2+60(Ts)^3+10(Ts)^4+(Ts)^5}$
3/5	$\frac{6720-2520(Ts)+360(Ts)^2-20(Ts)^3}{720+600(Ts)+240(Ts)^2+60(Ts)^3+10(Ts)^4+(Ts)^5}$
4/5	$\frac{15120-6720(Ts)+1260(Ts)^2-120(Ts)^3+5(Ts)^4}{720+600(Ts)+240(Ts)^2+60(Ts)^3+10(Ts)^4+(Ts)^5}$

Padé approximation of time delay transfer functions meets the weak conditions of physical realizability and introduces unstable zeroes into the transfer function. It is recommended to use it for $n \leq 10$ only. Their poles are stable for all practically usable orders and have a tendency to group to each other. The main advantage is the fact that there are no restrictions on the degrees of both polynomials. The Padé approximations of the time delay transfer function are sufficiently precise for the polynomials of different degrees. It is possible to meet in a natural way the both requirements on the approximations: similar behaviour in the right neighbourhood of the origin and sufficiently accurate behaviour for higher values of time (see Fig. 3 and 4, Tab. 6 and 7).

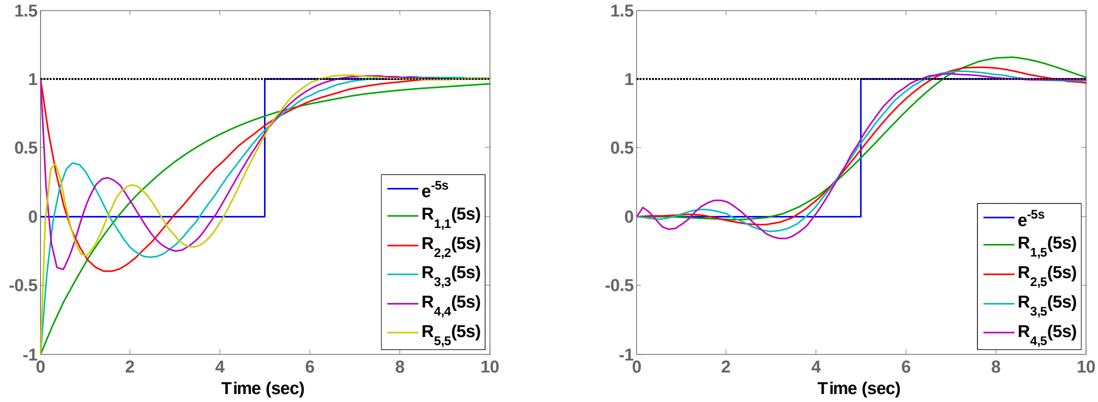

Figure 3: Padé approximations with same and different degrees of polynomials for the pure time delay

Table 6: COMPARISON OF ERRORS FOR PADÉ APPROXIMATION OF TIME DELAY TRANSFER FUNCTION WITH SAME DEGREES OF BOTH POLYNOMIALS

Padé approximation degrees	1/1	2/2	3/3	4/4	5/5
Pure time delay	1.3514	0.7710	0.5349	0.4080	0.3290
3rd order system with time delay	0.4444	0.1100	0.0334	0.0116	0.0045

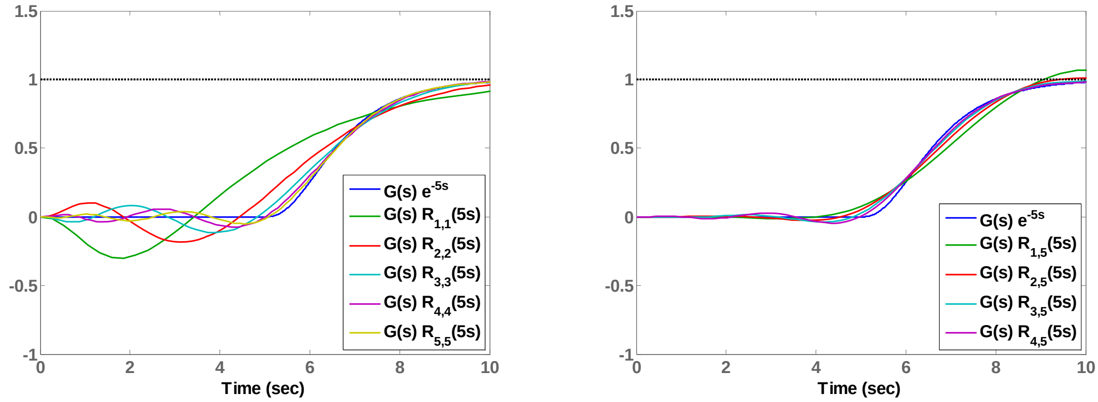

Figure 4: Padé approximations with same and different degrees of polynomials for the third order system with time delay

Table 7: COMPARISON OF ERRORS FOR PADÉ APPROXIMATION OF TIME DELAY TRANSFER FUNCTION WITH DIFFERENT DEGREES OF BOTH POLYNOMIALS

Padé approximation degrees	1/5	2/5	3/5	4/5
Pure time delay	0.3149	0.2288	0.2006	0.2025
3rd order system with time delay	0.0324	0.0124	0.0064	0.0046

3 Conclusions

A comparison of basic methods of rational approximation of time delay transfer functions is carried out in the contribution. Their advantages and disadvantages are briefly presented. The comparison of the approximations with the exact time delay is given in tables with numerical results and in figures with displayed time behaviour of the approximations. The integral of square error calculated using trapezoidal rule is used as the measure of their accuracy. The best results are given by Padé approximation $R_{m,n}(Ts)$ with different degrees ($m < n$) of the numerator and denominator.

References

[1] G. H. Golub, Ch. F. van Loan: *Matrix Computations*. Johns Hopkins University Press, Baltimore, Maryland, 1996

[2] B. C. Kuo. *Automatic Control Systems*. Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[3] A. Ralston: *Základy numerické matematiky*. Academia, Praha, 1973.

[4] M. Vajta. Some Remarks on Padé-Approximations [on line]. In: *Proceedings of the 3rd TEMPUS INTCOM Symposium on Intelligent Systems in Control and Measurement*, 53-58. 9.-14.9.2000, Veszprém, Hungary. [cited: 25.10.2009]. Accessible at: http://wwwhome.math.utwente.nl/~vajtam/publications/temp00-pade.pdf

[5] Wikipedia. The free Encyclopedia. *Padé Approximant* [on line]. [cited: 25.10.2009]. Accessible at: http://en.wikipedia.org/wiki/Pad%C3%A9_approximant

[6] Wolfram MathWorld. *Padé Approximant* [on line]. [cited: 25.10.2009]. Accessible at: http://mathworld.wolfram.com/PadeApproximant.html.

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