EVALUATION OF A CLAMPED-CLAMPED BEAM DEFLECTION DUE TO MAGNETIC FORCE

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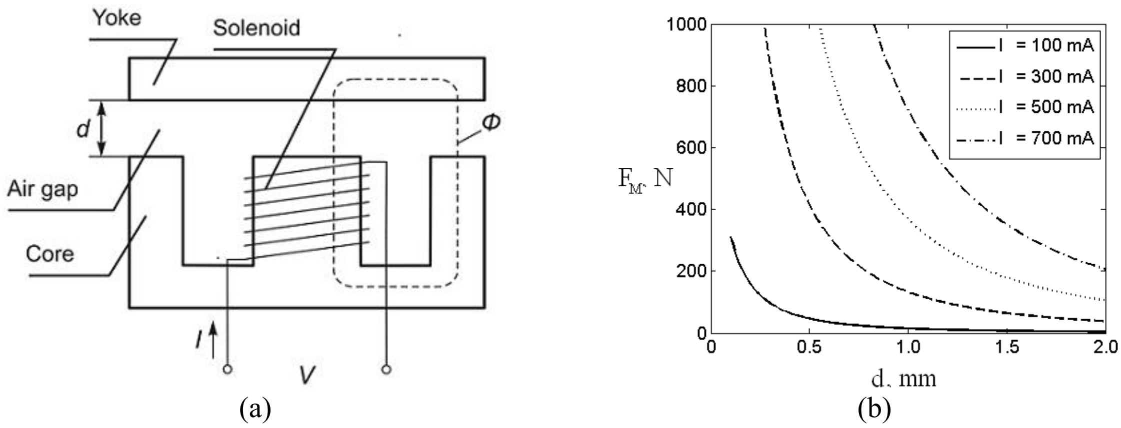
Abstract

In the article, a mechatronic system composed of a slender beam, subjected to a static magnetic force, is analyzed. Dependence of the magnetic force on the air gap width *d*, which is a function of the beam deflection, is described by the inverse square law 1/*d*^{2}. Action of the magnetic force causes bending of the beam. If its intensity surpasses certain limit the beam is buckled. The mathematical model of the magnetic force can be approximated by a polynomial expansion. It is of interest to analyse the nature of bending and limits of the linear approximation, as well as the limits leading to the buckled state in a generalised way. The cubic algebraic equation, describing the static equilibrium condition, is derived and solved in exact and in an approximate way using MATLAB® environment. The difference between exact and approximate solution is described, taking into consideration the physical background. Practical conclusions on use of the approximate solution are drawn in the conclusion.

1 Introduction

In some mechatronic applications a slender beam or plate, rigidly fixed on its boundaries is subjected to a magnetic force, generated between an energised solenoid with a ferromagnetic core and a ferromagnetic yoke fixed to the beam (Fig. 1(a)) [1-3]. Dependence of the force on the distance d, Fig. 1 (b), is described by the inverse square law: $1/d^2$.

The contribution presents discussion on evaluation of static deflection of such a mechatronic system, exposed to static magnetic force F_M. As shown in Fig. 2, the magnetic force is acting in the middle of the beam of length L and induces a deflection z_{max}. If intensity of the magneto-static force F_M surpasses certain limit the beam is buckled and attracted to the end-stops (see e.g. [1], p. 7-10).



(a) (b)

Figure 1: (a) scheme of an electromagnet; (b) Diagram of magnetic force F_M vs. displacement d [4]

2 Mathematical Model of the Equilibrium State

Let us consider a long slender beam, rigidly fixed on its both ends to the supports, with a yoke of mass m located in the midpoint (Fig. 2). Let assume, that the length L_m of the yoke is negligible in respect to the overall beam length L (i.e. $L_m << L$). Then the deflection, z_{max} at the beam midpoint due to the magnetic force F_M acting downward and localized at midpoint of the beam is [5]:

$$z_{\max}(F_M) = \frac{F_M}{192} \cdot \frac{L^3}{E_b I_b}, \tag{1}$$

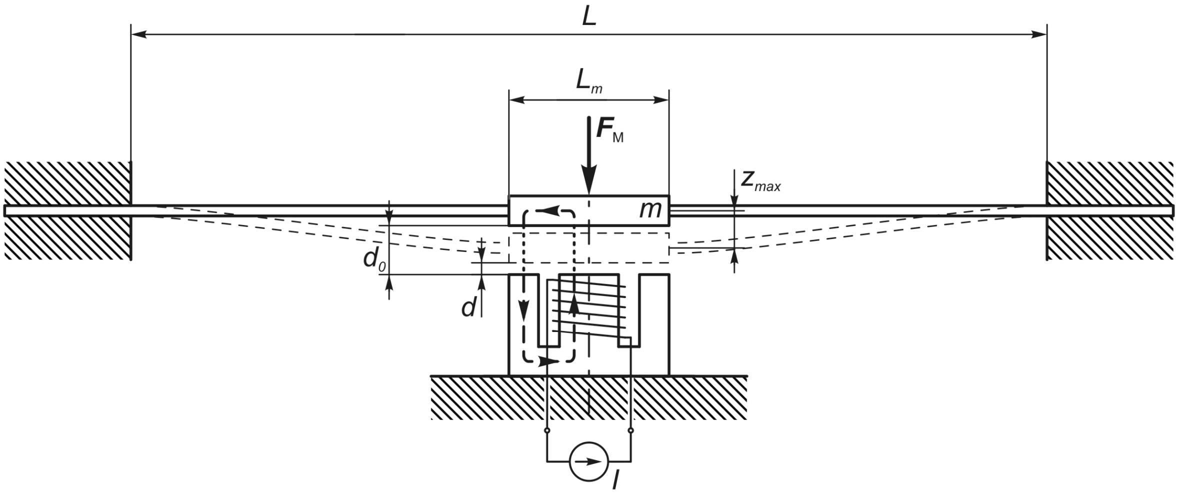


Figure 2: Schematics of the clamped-clamped beam with electromagnet
(flux line is denoted dashed).

where E_b is the modulus of elasticity (Young's module) of the beam material and I_b is the second moment of inertia of the beam. For a beam with a rectangular cross-section, $I_b = 1/12 \cdot bh^3$, where b is the width of the beam and h is its height. This can be expressed as $F_M = z_{max} \cdot k_{ef}$, where the theoretical value of the effective (lumped) stiffness k_{ef} of the clamped-clamped slender beam loaded at the midpoint is given in technical handbooks (e.g. [5]) as $k_{ef} = 192 \cdot (E_b I_b)/L^3$.

Energizing the electromagnet with a steady state (DC) current the magnitude of magneto-static force F_M can be described by a scalar equation [1-4]:

$$F_{\mathrm{M}}(d,I) = \frac{\mu_0 S N^2 I^2}{\left(2d + l_\Phi/\mu_{\mathrm{r}}\right)^2} \,. \tag{2}$$

The magnetic flux line is crossing twice the air gap as shown in Fig. 1 (a); μ_0 is the permeability of air and l_Φ is the flux line length in the ferromagnetic material of the relative permeability μ_r. In following the equivalent magnetic flux line length in the air, $d_{\mathrm{Fe}} = \frac{1}{2} l_\Phi / \mu_{\mathrm{r}}$ will be used. All parameters are either known from vendor's data, were measured experimentally or calculated from set-up geometry [2, 4].

The state, when the magnetic force $F_M(d, I)$ is in static equilibrium with the elastic force due to the beam deflection $z_{max} \cdot k_{ef}$ is described by Eq. (3). From the set-up geometry (Fig. 2) follows: $z_{max} = d_0 - d$, where d_0 is the initial distance between electromagnet and the beam in de-energised state:

$$(d_0 - d) k_{\mathrm{ef}} = F_{\mathrm{M}}(d, I) \,. \tag{3}$$

Let's introduce a non-dimensional air gap width α: $\alpha = (d_0 - d)/d_0$. (4)

From physical point of view, the quantity α is non-negative and cannot be larger than one. If $\alpha = 1$, beam is fully attracted by the electromagnet and adheres to its poles.

Introducing α into Eq. (3) and using Eq. (2), the equilibrium equation is (where $\delta_M = d_{Fe}/d_0$):

$$\alpha \cdot k_{ef} = \frac{F_{\mathrm{M}}}{d_0} = \left(\frac{\mu_0 S N^2}{4 d_0} \right) \cdot \frac{I^2}{\left[\left(d_0 - \alpha d_0 + d_{\mathrm{Fe}} \right) \right]^2} = \left(\frac{\mu_0 S N^2}{4 d_0} \right) \cdot \frac{I^2}{d_0^2 \left[\left(1 + \delta_{\mathrm{M}} \right) - \alpha \right]^2} \,. \tag{5}$$

Note, that material of electromagnet is assumed to be linear (no saturation) and neither fringing effects nor hysteretic effects are considered for simplicity.

Eq. (5) can be re-written as follows:

$$\alpha(I) \cdot k_{\mathrm{ef}} = \left(\frac{\mu_0 S N^2}{4 d_0^3} \right) \cdot \frac{I^2}{\left[\left(1 + \delta_{\mathrm{M}} \right) - \alpha(I) \right]^2} = K_{\mathrm{M}} \frac{I^2}{\left[\left(1 + \delta_{\mathrm{M}} \right) - \alpha(I) \right]^2} \,, \tag{6}$$

where the constant K_M represents the parameters of the electromagnet at the initial position d_0 of the de-energised system.

3 Solution of the Equilibrium Equation

The Eq. (6) can be solved for variable $\alpha(I)$ by an approximate approach using linear approximation, or in the exact way, applying analytical or numerical tools.

3.1 Approximate Solution

Let us assume that the condition $\alpha << 1$ is fulfilled. The denominator of the right hand side of Eq. (6) can be approximated by a McLaurin's series [6, 7]:

$$\alpha k_{ef} = K_{\mathrm{M}} I^2 \cdot \left\{ \frac{1}{(1+\delta_{\mathrm{M}})^2} + \frac{2}{(1+\delta_{\mathrm{M}})^3} \cdot \alpha + \ldots \right\}. \tag{7}$$

Just the first two terms of the expansion are considered, i.e. the linear approximation is used. After some algebra, noting that $0 \le \alpha \le 1$, formula for approximate calculation of α' emerges:

$$\alpha' = \frac{(1+\delta_{\mathrm{M}}) \cdot K_{\mathrm{M}} I^2}{k_{\mathrm{ef}} (1+\delta_{\mathrm{M}})^3 - 2 \cdot K_{\mathrm{M}} I^2}. \tag{8}$$

3.2 Exact Solution

Rewriting Eq. (6), a cubic equation is obtained:

$$\alpha \cdot [\alpha - (1+\delta_{\mathrm{M}})]^2 = \left(\frac{K_{\mathrm{M}}}{k_{\mathrm{ef}}} \right) I^2 \ , \tag{9a}$$

$$\alpha^3 - 2\alpha^2 (1+\delta_{\mathrm{M}}) + \alpha (1+\delta_{\mathrm{M}})^2 - (K_{\mathrm{M}}/k_{\mathrm{ef}}) I^2 = 0 \ . \tag{9b}$$

The solution of Eq. (9b) calls for the use of Cardano's formulas for evaluation of cubic equations [6, 7] or rely on numerical solvers of algebraic equations, embedded in simulation programming environment, e.g. MATLAB®. The numerical solution leads generally to three different complex roots. A detailed analysis according to [6, 7] revealed, that in analogy to the quadratic equation there is a cubic discriminator D_3, furnishing for $D_3 > 0$ three real roots. This is the case here. By further analysis two pairs of special real solutions of this cubic equation were found:

- a pair for $\alpha = 0$ and $\alpha = (1 + \delta_{\mathrm{M}}) > 1$, which is a result for $I = 0$;
- a pair for $\alpha = (1 + \delta_{\mathrm{M}})/3$, $\alpha = 4(1 + \delta_{\mathrm{M}})/3$, which results if I attains a specific value I_{crit}:

$$I_{\mathrm{crit}}^2 = \frac{4}{27} (1+\delta_{\mathrm{M}})^3 \left(\frac{k_{\mathrm{ef}}}{K_{\mathrm{M}}} \right). \tag{10}$$

Note, that the critical current I_{crit} is determined by the beam stiffness k_{ef}, the electromagnetic properties K_{M}, and the ratio $\delta_{\mathrm{M}} = d_{\mathrm{Fe}}/d_0$. This ratio can be interpreted as the ratio of the magnetic circuit equivalent flux line length in the air to the initial air gap width d_0. The value $\alpha = 4(1 + \delta_{\mathrm{M}})/3$ corresponds to a *triple real root* at $D_3 = 0$ (Fig. 3). For $I > I_{\mathrm{crit}}$ there is only a single real root and two complex conjugate roots.

3.3 Comparison of Approximate and Exact Solution

Let us introduce a normalized current I_{N}, i.e. the current normalized by the value of critical current, $I_{\mathrm{N}} = I/I_{\mathrm{crit}}$. Then Eqs. (9) can be re-formulated in a following way:

$$\alpha' = \frac{(1+\delta_{\mathrm{M}})}{\left(\frac{27}{4} \right) \frac{1}{I_N^2} - 2} \ , \tag{11a}$$

$$\alpha^3 - 2\alpha^2 (1+\delta_{\mathrm{M}}) + \alpha (1+\delta_{\mathrm{M}})^2 - \frac{4}{27} I_N^2 (1+\delta_{\mathrm{M}})^3 = 0 \ . \tag{11b}$$

The numerical evaluation of Eq. (11b) was done in MATLAB® environment. For calculation the exact solutions α the MATLAB® function 'roots' is applied, returning a complex three element vector for each I_{N} value, incremented in steps of the order of 2×10^{-3}. Then the roots for each I_{N} value are ordered in ascending order, depending on the number of non-zero real roots. The results are finally plotted in the form of line graphs.

The dependence of the exact solution α in respect of the normalised current I_N is plotted in Fig. 3. Note, that this is not a plot of a function, because for any positive value of $I_N < 1$ three mutually different values are possible, because $D_3 > 0$. This is just the graphical representation of the course of real roots of the cubic Eq. (11b) without any relation to the underlying physical background. Moreover, the course of approximate solution α' (Eq. (11a)) is plotted in dashed, too. Note, that the approximate solution α' is valid under the above assumption of $\alpha << 1$.

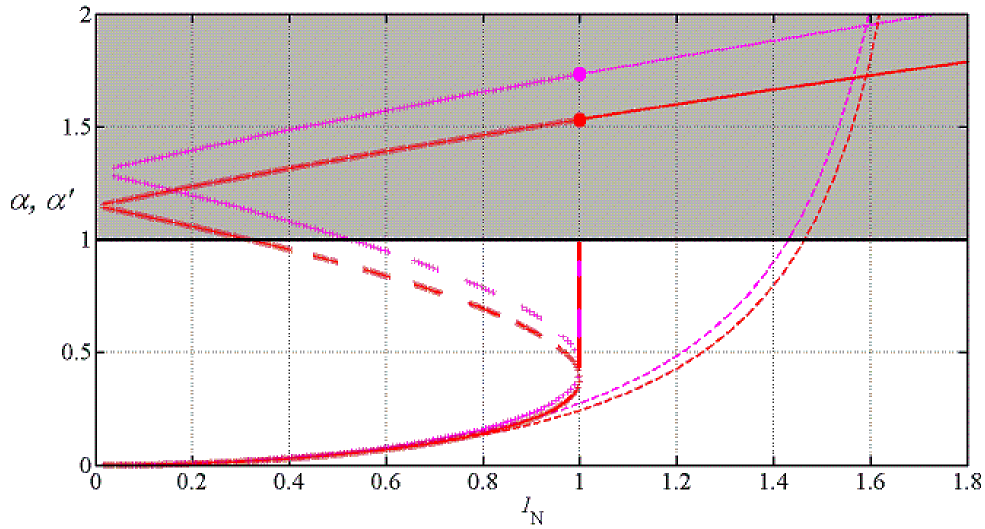


Figure 3: Courses of the exact solution α (thick line) and the approximate solution α' (thin line) for $d_{Fe} = 0.15$ mm and $d_0 = 0.5$ mm (red), $d_0 = 1.0$ mm (blue); solution for $D_3 = 0$ are highlighted

3.4 Physical Interpretation of the Results

As stated above, the physically feasible values of numerical solution of the cubic Eq. (11b) are bound to the interval [0, 1] (white area in Fig. 3); hence the solutions above the bold limit for $\alpha = 1$(in the gray area) have no physical meaning (the beam would have to move within the electromagnetic core!). So, only values of $\alpha \leq 1$ are meaningful.

i. The dashed course commencing at $\alpha = 1$ is also not realistic either, because this would assume that the elastic beam was buckled prior to energising the field. Hence, this branch has no physical meaning in this case either.

ii. The physically plausible course is the lowest curve, starting at zero and reaching for $I = I_{crit}$ the value of $\alpha = (1 + \delta_M)/3$ (Fig. 3). However, for the value I_{crit} $(I_N = 1)$ two different solutions do exist: $\alpha = (1 + \delta_M)/3$ and $\alpha = 4(1 + \delta_M)/3$. This can by interpreted as the limit of stability of the physical system: **at the critical current the beam buckles from the value of $\alpha = (1 + \delta_M)/3$ to α=1 in a jump.**

iii. If the current would revert from a value of $I > I_{crit}$ the beam would follow the same trajectory, i.e. as soon as the value of I_N drops below unity value the beam, firstly adhering to the magnet core, would attain (after extinction of the transient phenomenon) a position corresponding to the $\alpha = (1 + \delta_M)/3$.

The critical value of the magnetising current I, I_{crit} is crucial for discrimination between bending and buckling of the elastic slender beam rigidly fixed on both ends. Moreover, **the maximal displacement due to bending prior to transition to the buckled state**, in the view of the substitution introduced by Eq. (6) is:

$$d_{lim} = \frac{d_0}{3}(2 - \delta_M) = (4d_0 - l_\Phi / \mu_r)/6 \ , \tag{12}$$

which, as seen, depends on the ratio $\delta_M = d_{Fe}/d_0$ and the de-energised distance d_0.

Next, the applicability of the linear approximation has to be assessed. From Fig. 3 it is seen that when $I_N < 0.80$ there is no visible difference between the exact solution and the approximate solution. To be more precise, for the de-energised air-gap width $d_0 = 1.0$ mm and corresponding $\delta_M = 0.15$ for

$I_N = 0.80$ the difference between the exact solution $\alpha = 0.141$ and the approximate solution $\alpha' = 0.135$ is some -5 %, i.e. still technically acceptable. For $I_N = 0.50$ the difference between the exact solution $\alpha = 0.0463$ and the approximate one $\alpha' = 0.0460$ is less than -0.6 %. So, these are the practical limits of applicability of the approximate solution.

4 Comparison of Experimental and Simulation Results

The authors have experience with an experimental set-up (Fig. 4), which consists of an aluminium beam (i.e. non-ferromagnetic material) of rectangular cross-section. The beam is thoroughly fixed to robust supports to ensure clamped boundary conditions. In the middle of the beam a variable dead load is mounted. From below the yoke, matched to the electromagnet, is fixed. The electromagnet used is of industrial type, rated at 700 mA reaching maximum holding force at zero air gap of 3.0 kN. The force-distance characteristics according to Eq. (2) were measured for various current intensities experimentally [4] (Fig. 1(b)). From electromagnet vendor's data sheet and by measuring relevant properties all necessary dimensions, required in simulations, were obtained [4].

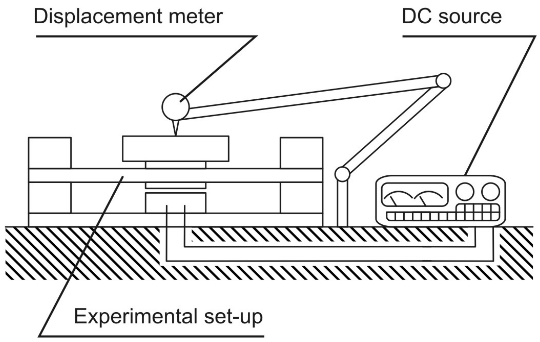


Figure 4: Schematics of the measurement setup

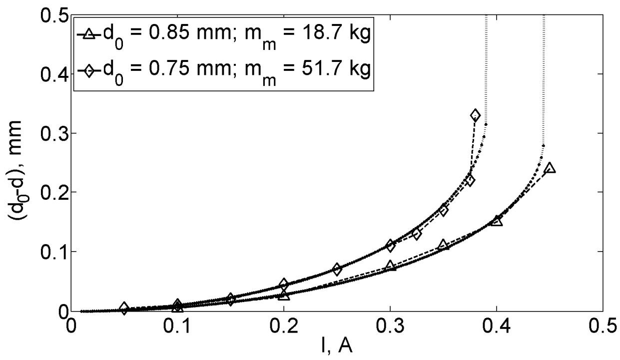


Figure 5: Comparison of simulated (solid) and measured (dashed with discrete points) courses

A dial displacement indicator was used for measuring the beam static deflection. In Fig. 5, two different distances d_0 in de-energised condition were pre-set, namely, 0.75 mm, 0.85 mm. In each of these two cases the beam was loaded by a different dead mass inclusive yoke (in turn 51.7 kg, 18.7 kg). The electromagnet was driven from a regulated DC source, while the magnetising current I was varied in steps between zero and 0.5 A. The measurements were conducted in such way, that the limit of Eq. (11) was not surpassed. The experimental results shown in Fig. 5 (after [4]) are compared to the theoretical curves for the exact solution of the Eq. (11b). Moreover, the effective beam stiffness k_{ef} was tuned experimentally and slightly varies on dead mass because of non-ideal boundary conditions. The limiting currents, calculated according to Eq. (10), were 0.39 A and 0.46 A, respectively. As can be seen from Fig. 5, this corresponds well with the measured courses.

Note the observed attempt to buckling in the experimental data for $d_0 = 0.75$ mm, underlining the correctness of the exact approach. Note also the good agreement of the experimental data and the exact solution.

5 Practical Conclusions

The practical lesson from this study is following:

- There is a critical current I_{crit}, whose magnitude is given by Eq. (10). If the solenoid is energised by a larger current the elastic beam buckles and is permanently attracted to the electromagnet.
- This happens at approximately one third of the static air gap width. More precisely this margin is given by Eq. (12) and depends on the ratio of the equivalent flux line length in the air of the magnetic circuit to the initial air gap width.
- The description of the elastic and magnetic forces equilibrium by an approximate description (linearised form) of the magnetic force (Eq. (8) or (11a)) can be used in the current range of approximately 80 % of the critical current.

Acknowledgement: This paper is a result of project No. 2/0075/10 of the VEGA Grant Agency.

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