COMPARISON OF EULER'S- AND TAYLOR'S EXPANSION METHODS FOR NUMERICAL SOLUTION OF NON-LINEAR SYSTEM OF DIFFERENTIAL EQUATION

Branislav Dobrucký, Mariana Beňová, Daniela Gombárska

Faculty of Electrical Engineering, University of Žilina, Slovak Republic

Abstract

The paper deals with Euler's- and Taylor's expansion methods for next numerical solution in Matlab environment. There are many applications in technical practise described and modelled by linear or non-linear differential equation (DE) systems. A fictitious exciting functions method makes possible numerical solution of this DE system with non-stationary matrices. The solution of examples with non-linear inductance is presented as well in the paper.

1 ODE System in Matrix State-Space Form

There are many applications in technical practise modelled linear or non-linear differential equation (DE) systems. Let’s have system of two first order ODEs (which can be given/rewritten as one ODE of the 2nd order)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} - a_{11}x_1 - a_{12}x_2 = b_{11}u_1, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} - a_{21}x_1 - a_{22}x_2 = b_{22}.$$

The system can be also presented in matrix state-space form

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \bar{x} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \bar{u},$$

where (1)

$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ are system- and transition matrices,

$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ are state (state-variables)- and exciting vectors, respectively.

Such a linear system of ODE can be solved analytically and/or also numerically (e.g. by Euler explicit method) [1], [2], [4]. If the matrix elements are non-stationary (e.g. time dependent ones) then system of equations cannot be solved by the methods using the matrix operation as e.g. Euler implicit or Taylor expansion methods.

2 Principle of Fictitious Exciting Functions Method

If a_{11} and a_{12} elements of $\boldsymbol{A}$ matrix are non-stationary (e.g. time dependent ones) and b_{12}, b_{21}, b_{22} and $u_2 = 0$ then system of Eq. (1) can be rearranged into following form [3]

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix} \bar{x} + \begin{pmatrix} b_{11} & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ a_{11}x_1 \\ a_{12}x_2 \end{pmatrix},$$

where $\bar{u}_f = (u_1;\, a_{11}x_1;\, a_{12}x_2)^{\mathrm{T}}$ is **fictitious exciting vector** and $a_{11}x_1$; $a_{12}x_2$ are **fictitious exciting functions**,

$\boldsymbol{A}_f = \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}$ is modified (fictitious) state matrix,

$\boldsymbol{B}_f = \begin{pmatrix} b_{11} & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is modified (fictitious) transition matrix of the system.

Let’s consider Euler’s- and Taylor’s expansion methods for numerical solution of Eq. (.). We obtain

a) Euler explicit method yields

$$\bar{x}_{n+1} = (\boldsymbol{E} + h\boldsymbol{A}_f)\bar{x}_n + h\boldsymbol{B}_f\bar{u}_{fn}$$

where h is integration step;

$\boldsymbol{E}$ is unity matrix.

That method is sensitive on integration step. Stability condition is that h should be smaller than $2/|Re\{\lambda_i\}|_{max}$ [4].

b) Euler implicit method yields

$$\bar{x}_{n+1} = (\boldsymbol{E} - h\boldsymbol{A}_f)^{-1}[\bar{x}_n + h\boldsymbol{B}_f\bar{u}_{fn}]$$

where $\boldsymbol{F} = (\boldsymbol{E} - h\boldsymbol{A}_f)^{-1}$ is fundamental matrix of the system. Contrary to above this method is for negative real part of eigenvalues absolutely stable (A-stabile) for any positive step h [4].

c) Taylor expansion yields [4]

$$\boldsymbol{F} = \exp(\boldsymbol{A}h) = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n h^n}{n!}$$

and similarly

$$\boldsymbol{F}_{n+1} = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^{n+1} h^{n+1}}{(n+1)!}; \quad \boldsymbol{G} = \boldsymbol{A}^{-1}\boldsymbol{F}_{\boldsymbol{n+1}}\boldsymbol{B}$$

So, choosing appropriated number of series member n one can obtain

$$\bar{x}_{n+1} = \boldsymbol{F}\bar{x}_n + \boldsymbol{G}\bar{u}_{fn}$$

The method is similarly to Euler implicit above also A-stabile one.

All discrete equations carried-out by Euler explicit-, implicit- and Taylor expansion methods are easily solvable by numerical computing because their modified (fictitious) matrices are stationary ones.

3 Application of all three method for 2nd order electric circuit solving

2nd order electric circuit with non-linear element L_{non} is depicted in Fig. 1

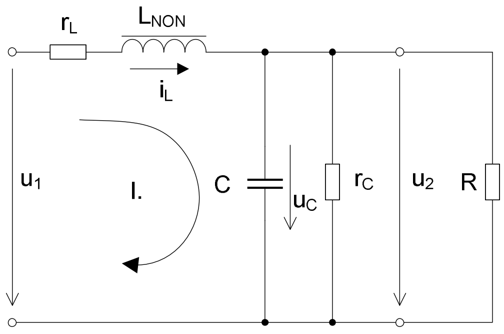


Figure 1: 2nd order electric circuit with non-linear element $L_{non} = \mathrm{f}(i_L)$

The circuit can be described by two first order ODEs system as follow

$$\frac{\mathrm{d}i_L}{\mathrm{d}t} - a_{11}i_L - a_{12}u_C = b_{11}u_1$$

$$\frac{\mathrm{d}u_C}{\mathrm{d}t} - a_{21}i_L - a_{22}u_C = 0$$

where: $a_{11} = -\frac{r_L}{L_{non}} = -\frac{1}{\tau_1},\ a_{12} = -b_{11} = -\frac{1}{L_{non}},\ a_{21} = \frac{1}{C},\ a_{22} = -\left(\frac{1}{r_C} + \frac{1}{R}\right)\frac{1}{C} = -\frac{1}{\tau_2}$

The dynamical inductance L_{non} is non-linear function of the inductor current

$$L_{non} = \mathrm{f}(i_L)$$

The functional dependency can be obtained directly from *B-H* characteristic of magnetic core of the inductor [6], [7], by measurement or using various functional linearized substitutions. For SIFERRIT U60 material [6] is that dependency shown in Fig. 2 with other functional linearized dependencies.

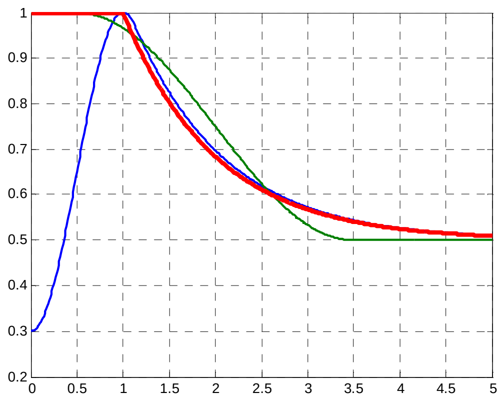


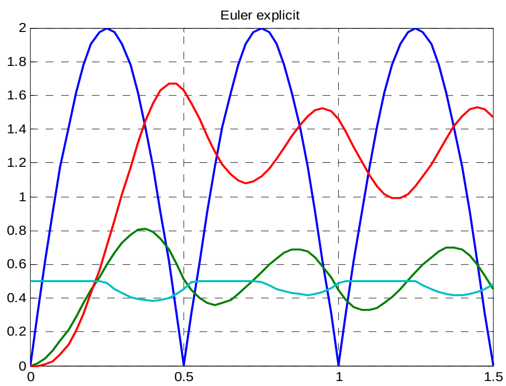
Figure 2: Non-linear dependency $L_{non} = \mathrm{f}(i_L)$ for U 60 material [6] (a) and other functional linearized dependencies (b) in p.u. where x: i_L and y: L_{non}

So, $L_{non} = \mathrm{f}(i_L)$ can be expressed by some different models:

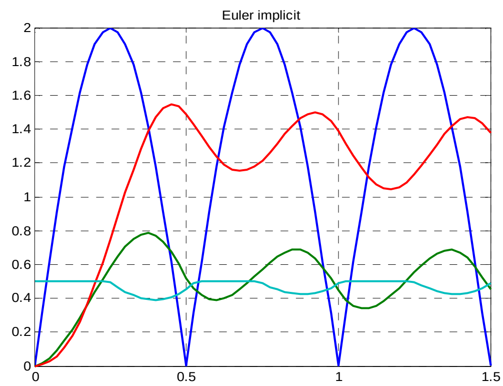
Linearized model I.: if $i_L < I_{Lnom}$ then $L_{non} = L_{lin}$ else

$$L_{non} = (L_{lin} - L_\infty)\exp\left[-{}^{1}\!/_{\tau}\,.\,(i_L - I_{Llin})\right] + L_\infty,$$

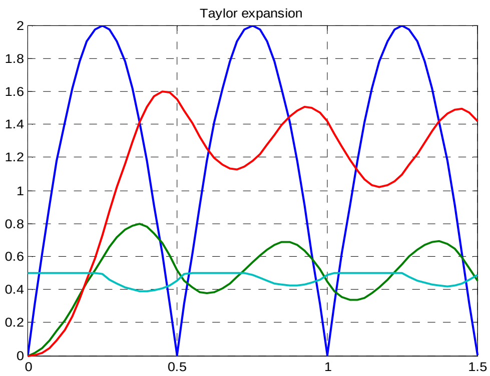
where $L_\infty = 0.5\,L_{lin}$ as given in Fig. 2. This model has been used for simulation experiments shown in the F Fig. 3a,b,c. Other models are referred in [5].



a)



b)



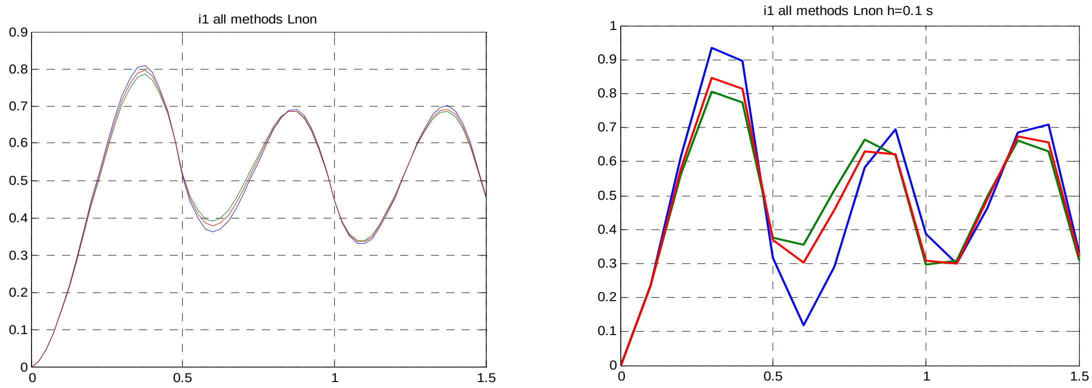
c)

Figure 3: Time dependences of input voltage and state variables i_L, u_C and L_{non} of the circuit considering $L_{non} = \mathrm{f}(i_L)$:

a) Euler explicit method
b) Euler implicit method
c) Taylor's expansion

Parameters: L_{lin} = 0.5; $C = 2/\omega^2$; R = 2.5 Ω; r1=r2=1e-2; U=2; 1/τ = 2.0; I1lin = 0.5; h=2.5e-2; T=1; $\omega = 2\pi/T$;

Comparison of all three computing methods is shown in next Fig. 4a,b.



a) b)

Figure 4: Time dependences of input voltage and state variables i_L, u_C and L_{non} at different ratio of h=0.025 s a) and h=0.1 s b)

The average value of input voltage (regarding to sinusoidal shape) is

$$U_{1AV} = \frac{2}{\pi}U = \frac{2}{\pi}2 = 1.2732 \text{ V}$$

Then average value of the input (= inductor) current is in steady-state

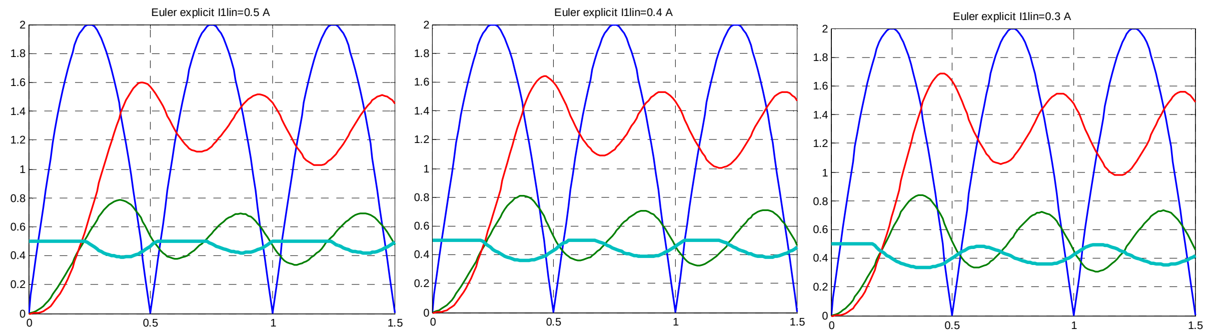
$$I_{1AV} = \frac{U_{AV}}{r_L + R} = \frac{2}{0.01 + 2.5} = 0.5052 \text{ A},$$

which is taken as nominal one (I_{Lnom}) and therefore is the inductor current compared with that value at non-linear inductor model.

The average value of output voltage is then

$$U_{2AV} = U_{1AV} - r_L.I_{1AV} = \frac{2}{\pi}U - r_L.\frac{U_{AV}}{r_L + R} = \frac{2}{\pi}2 - 0.01\frac{2}{0.01 + 2.5} = 1.2682 \text{ V}.$$

Taking into account different ratio of the I_{Llin} and I_{Lnom} then for 100-, 80- and 60 % one obtains the time waveforms as shown in Fig. 5a,b,c



a) b) c)

Figure 5: Time dependences of input voltage and state variables i_L, u_C and L_{non} at different ratio of $^{I_{Llin}}/_{I_{Lnom}}$ equal 1.0 a), 0.8 b) and 0.6 c)

4 Conclusion, discussion

Comparison of Euler's- and Taylor's expansion methods has been given using computation of the 2nd order non-linear ODE system

The following conclusion resulting from the simulation experiments given in Figs. 3a,b,c, 4a,b and 5a,b,c:

Computational results of non-linear ODE system solving by three independent methods: Euler explicit/ and implicit ones and Taylor's expansion are very similar at sufficiently small integration step smaller the 0.025 s.

Euler explicit method shows rather big error at integration step h=0.1 s (Fig. 4b).

Given non-linearity of the inductor L=f(i_L) causes the increasing of the input current (up to 5 %) and output voltage (up to 10 %) at different ratio of ${I_{Llin}}/{I_{Lnom}}$ (1.0; 0.8; 0.6), Figs. 5a,b,c.

Acknowledgement

The authors wish to thank for the financial support to the VEGA Agency for the project No. 1/0470/09.

References

[1] J. Skrášek, Z.Tichý, *Fundamentals of Applied Mathematics II* (in Czech). SNTL Publisher, Prague (CZ), 1986.

[2] D. Mayer, B. Ulrych: *Numerical Analysis of Non-Linear Electrical Circuits by State Variables Method* (in Czech). Electrical Review (Elektrotechnický obzor), Vol. 65, 1976, No. 1, pp. 30-38.

[3] L Nevařil: *State Variables on HP 9830 Desk Computer* (in Czech). Research Report of EVPU Research Institute, N. Dubnica (SK), No. V-0516, 1978

[4] H. Mann: *Using Computer for Electrical Designs* (in Czech). SNTL/ALFA Publisher, Prague (CZ)/Bratislava (SK), 1984.

[5] M. Beňová, B. Dobrucký, D. Gombárska, R. Radvan: *Using Fictitious Exciting Functions Method for Non-Linear Differential Equation System Solving*. e-Proc. of ICPM'11 Int'l Conf. on Presentation of Mathematics, Liberec (CZ), pp. CD-ROM, Nov 2011.

[6] - : *Ferrites – Soft Magnetic Materials.* Siemens AG Data Book, Muenchen (DE), 1979 /80, p. 23, ISBN X-X.

[7] R. Radvan, Dobrucký, B., Frivaldský, M., Rafajdus, P.: *Modelling and Design of HF 200 kHz Transformers for Hard- and Soft- Switching Application.* Electronics and Electrical Engineering, Vol. 110, Nr. 4, pp. 7-12, 2011.

[8] B. Dobrucký, M. Marčoková, M. Beňová, M. Lusková: *Using Theory of Series for Distortion Determination of Non-Harmonic Quantities of Power Electronic Systems*. Proc. of MME'10 Int'l Symposium on Mathematical Methods in Engineering, Coimbra (PT), pp. 57-66, Oct 2010.

Branislav Dobrucký
Department of Mechatronics and Electronics, dobrucky@fel.uniza.sk, phone +421 41 513 1602

Mariana Beňová
Department of Electromagnetic and Biomedical Engineering, benova@fel.uniza.sk, phone +421 41 513 5090

Daniela Gombárska
Department of Electromagnetic and Biomedical Engineering, gombarska@fel.uniza.sk, phone +421 41 513 5090