COMPARISON OF EULER'S- AND TAYLOR'S EXPANSION METHODS FOR NUMERICAL SOLUTION OF NON-LINEAR SYSTEM OF DIFFERENTIAL EQUATION

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Abstract
The paper deals with Euler’s- and Taylor's expansion methods for numerical solution in Matlab environment. There are many applications in technical practise described and modelled by linear or non-linear differential equation (DE) systems. A fictitious exciting functions method makes possible numerical solution of this DE system with non-stationary matrices. The solution of examples with non-linear inductance is presented as well in the paper.

1 ODE System in Matrix State-Space Form
There are many applications in technical practise modelled linear or non-linear differential equation (DE) systems. Let’s have system of two first order ODEs (which can be given/rewritten as one ODE of the 2nd order)
\[
\frac{dx_1}{dt} - a_{11}x_1 - a_{12}x_2 = b_{11}u_1, \quad \frac{dx_2}{dt} - a_{21}x_1 - a_{22}x_2 = b_{22}.
\]
The system can be also presented in matrix state-space form
\[
\frac{d\bar{x}}{dt} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \bar{x} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \bar{u},
\]
where
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\]
are system- and transition matrices,
\[
\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]
are state (state-variables)- and exciting vectors, respectively.
Such a linear system of ODE can be solved analytically and/or also numerically (e.g. by Euler explicit method) [1], [2], [4]. If the matrix elements are non-stationary (e.g. time dependent ones) then system of equations cannot be solved by the methods using the matrix operation as e.g. Euler implicit or Taylor expansion methods.

2 Principle of Fictitious Exciting Functions Method
If \(a_{11}\) and \(a_{12}\) elements of \(A\) matrix are non-stationary (e.g. time dependent ones) and \(b_{12}, b_{21}, b_{22}\) and \(u_2 = 0\) then system of Eq. (1) can be rearranged into following form [3]
\[
\frac{d\bar{x}}{dt} = \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix} \bar{x} + \begin{pmatrix} b_{11} & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ a_{11}x_1 \\ a_{12}x_2 \end{pmatrix},
\]
where \(\bar{u}_f = (u_1; a_{11}x_1; a_{12}x_2)^T\) is \textbf{fictitious exciting vector} and \(a_{11}x_1; a_{12}x_2\) are \textbf{fictitious exciting functions},
\[
A_f = \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}
\]
is modified (fictitious) state matrix,
\[
B_f = \begin{pmatrix} b_{11} & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]
is modified (fictitious) transition matrix of the system.

Let’s consider Euler’s- and Taylor’s expansion methods for numerical solution of Eq. (.). We obtain
a) Euler explicit method yields
\[ \tilde{x}_{n+1} = (E + hA_f)\tilde{x}_n + hB_f\tilde{u}_{fn} \]
where \( h \) is integration step;
\( E \) is unity matrix.
That method is sensitive on integration step. Stability condition is that \( h \) should be smaller than \( 2/|\text{Re}\{\lambda_i\}|_{\text{max}} \) [4].
b) Euler implicit method yields
\[ \tilde{x}_{n+1} = (E - hA_f)^{-1}\left[\tilde{x}_n + hB_f\tilde{u}_{fn}\right] \]
where \( F = (E - hA_f)^{-1} \) is fundamental matrix of the system. Contrary to above this method is for negative real part of eigenvalues absolutely stable (A-stabile) for any positive step \( h \) [4].
c) Taylor expansion yields [4]
\[ F = \exp(Ah) = \sum_{n=0}^{\infty} \frac{A^n h^n}{n!} \]
and similarly
\[ F_{n+1} = \sum_{n=0}^{\infty} \frac{A^n h^{n+1}}{(n+1)!}; \quad G = A^{-1}F_{n+1}B \]
So, choosing appropriated number of series member \( n \) one can obtain
\[ \tilde{x}_{n+1} = F \tilde{x}_n + G\tilde{u}_{fn} \]
The method is similarly to Euler implicit above also A-stabile one.
All discrete equations carried-out by Euler explicit-, implicit- and Taylor expansion methods are easily solvable by numerical computing because their modified (fictitious) matrices are stationary ones.

3 Application of all three method for 2nd order electric circuit solving

2nd order electric circuit with non-linear element \( L_{non} \) is depicted in Fig. 1

![Figure 1: 2nd order electric circuit with non-linear element](image)

The circuit can be described by two first order ODEs system as follow
\[ \frac{di_l}{dt} - a_{11}i_l - a_{12}u_c = b_{11}u_1 \]
\[ \frac{du_c}{dt} - a_{21}i_l - a_{22}u_c = 0 \]
where: \( a_{11} = -\frac{r_l}{L_{non}} = -\frac{1}{\frac{1}{r_1}}, \ a_{12} = b_{11} = -\frac{1}{L_{non}}, \ a_{21} = \frac{1}{C}, \ a_{22} = -\left(\frac{1}{r_c} + \frac{1}{R}\right) = -\frac{1}{\frac{1}{r_2}} \)
The dynamical inductance $L_{non}$ is non-linear function of the inductor current

$$L_{non} = f(i_L)$$

The functional dependency can be obtained directly from $B-H$ characteristic of magnetic core of the inductor [6], [7], by measurement or using various functional linearized substitutions. For SIFERRIT U60 material [6] is that dependency shown in Fig. 2 with other functional linearized dependencies.

![Non-linear dependency](image)

Figure 2: Non-linear dependency $L_{non} = f(i_L)$ for U 60 material [6] (a) and other functional linearized dependencies (b) in p.u. where x: $i_L$ and y: $L_{non}$

So, $L_{non} = f(i_L)$ can be expressed by some different models:

Linearized model I.: if $i_L < L_{nom}$ then $L_{non} = L_{lin}$ else

$$L_{non} = (L_{lin} - L_\infty) \exp \left[ -\frac{1}{\tau} (i_L - L_{lin}) \right] + L_\infty,$$

where $L_\infty = 0.5 L_{lin}$ as given in Fig. 2. This model has been used for simulation experiments shown in the Fig. 3a,b,c. Other models are referred in [5].

![Time dependences](image)

Figure 3: Time dependences of input voltage and state variables $i_L$, $u_C$ and $L_{non}$ of the circuit considering $L_{non} = f(i_L)$:

a) Euler explicit method
b) Euler implicit method
c) Taylor’s expansion

Parameters: $L_{lin} = 0.5$; $C = 2/\omega^2$; $R = 2.5$ Ω; $r1=r2=1e-2$; $U=2$; $1/\tau = 2.0$; $I1lin = 0.5$; $h=2.5e-2$; $T=1$; $\omega = 2\pi / T$;
Comparison of all three computing methods is shown in next Fig. 4a,b.

![Figure 4: Time dependences of input voltage and state variables $i_L$, $u_C$ and $L_{non}$ at different ratio of $h=0.025$ s a) and $h=0.1$ s b)](image)

The average value of input voltage (regarding to sinusoidal shape) is

$$U_{1AV} = \frac{2}{\pi} U = \frac{2}{\pi} 2 = 1.2732 \text{ V}$$

Then average value of the input (= inductor) current is in steady-state

$$I_{1AV} = \frac{U_{AV}}{\tau_L + R} = \frac{2}{0.01 + 2.5} = 0.5052 \text{ A},$$

which is taken as nominal one ($I_{Lnom}$) and therefore is the inductor current compared with that value at non-linear inductor model.

The average value of output voltage is then

$$U_{2AV} = U_{1AV} - r_L I_{1AV} = \frac{2}{\pi} U - r_L \cdot \frac{2}{\pi} 2 - 0.01 \frac{2}{0.01 + 2.5} = 1.2682 \text{ V}.$$  

Taking into account different ratio of the $I_{Llin}$ and $I_{Lnom}$ then for 100-, 80- and 60 % one obtains the time waveforms as shown in Fig. 5a,b,c

![Figure 5: Time dependences of input voltage and state variables $i_L$, $u_C$ and $L_{non}$ at different ratio of $I_{Llin}$/$I_{Lnom}$ equal 1.0 a), 0.8 b) and 0.6 c)](image)

4 Conclusion, discussion

Comparison of Euler's- and Taylor's expansion methods has been given using computation of the 2nd order non-linear ODE system

The following conclusion resulting from the simulation experiments given in Figs. 3a,b,c, 4a,b and 5a,b,c:
Computational results of non-linear ODE system solving by three independent methods: Euler explicit/ and implicit ones and Taylor’s expansion are very similar at sufficiently small integration step smaller the 0.025 s.

Euler explicit method shows rather big error at integration step h=0.1 s (Fig. 4b).

Given non-linearity of the inductor $L=f(i_L)$ causes the increasing of the input current (up to 5 %) and output voltage (up to 10 %) at different ratio of $\frac{I_{L,\text{lin}}}{I_{L,\text{nom}}}$ (1.0; 0.8; 0.6), Figs. 5a,b,c.

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