DG METHOD FOR SCALAR LINEAR CONVECTION-DIFFUSION EQUATION

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Abstract

Among a wide class of numerical methods, the discontinuous Galerkin (DG) method represents a promising technique for the solution of convection-diffusion problems. The paper is devoted to the numerical solution of a scalar linear convection-diffusion equation. For the space discretization DG method is used and time is discretized with the aid of backward Euler method. The resulting scheme leads to a system of linear algebraic equations at each time step. Results of numerical experiments are presented.

1 Introduction

Our aim is to present a sufficiently robust, accurate and efficient numerical method for solution of convection-diffusion problems. As a model problem, we consider a one-dimensional scalar linear convection-diffusion equation. The discontinuous Galerkin (DG) methods have become very popular numerical technique for the solution of such problems. DG space semi-discretization uses higher order piecewise polynomial discontinuous approximation on arbitrary meshes, for a survey, see [2], [3]. Among several variants of DG methods we prefer the so-called interior penalty Galerkin (IPG) discretizations. We deal with three variants of IPG, namely nonsymmetric (NIPG), symmetric (SIPG) and incomplete interior penalty Galerkin (IIPG) techniques, see [1]. The discretization in time coordinate is performed with the aid of the backward Euler method, sidetracking the time step restriction well-known from the explicit schemes. Consequently, the fully discrete problem is represented by the system of algebraic equations. Within this paper we present the derivation of the whole scheme, from a continuous problem to the discrete one, and append the set of numerical experiments carried out in MATLAB, for more details see [6].

2 Problem formulation

We consider the following *unsteady linear 1D convection–diffusion problem:* Let $I \equiv (a, b) \subset I\!R$ be an open bounded interval and $T > 0$. We seek a function $u : Q_T = I \times (0, T) \to I\!R$ such that

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(c(x)\,u) = \frac{\partial}{\partial x}\left(\nu(x)\,\frac{\partial}{\partial x}u\right) + g \quad \text{in } Q_T, \tag{1}$$

$$u(a,t) = u_D^a(t) \quad \text{and} \quad u(b,t) = u_D^b(t), \quad t \in (0,T) \tag{2}$$

$$u(x,0) = u^0(x), \quad x \in I, \tag{3}$$

where $g : Q_T \to I\!R$ is the source term, $u_D^a, u_D^b : (0,T) \to I\!R$ are Dirichlet boundary conditions, $u^0 : I \to I\!R$ is the initial condition, $c(x) : I \to I\!R$ represents the convective part and $\nu(x) : I \to I\!R$ plays a role of diffusive coefficient. Moreover, we assume

$$\exists\, \nu_0,\, \nu_1 \in I\!R : 0 < \nu_0 \le \nu(x) \le \nu_1 \quad \forall\, x \in I. \tag{4}$$

The *initial boundary value* problem (1) – (3) is equipped with the initial condition (3) and the *Dirichlet* boundary conditions (2) prescribed at both boundary points of domain I but it is also possible to consider mixed Dirichlet–Neumann boundary conditions.

3 Discretization

Let $\mathcal{T}_h$ $(h > 0)$ be a family of the partitions of the closure $\overline{I} = [a, b]$ of the domain I into N closed mutually disjoint subintervals $I_k = [x_{k-1}, x_k]$ with length $h_k := x_k - x_{k-1}$ and the symbol $\mathcal{J}$ stands for an index set $\{1, \ldots, N\}$. Then we call $\mathcal{T}_h = \{I_k,\, k \in \mathcal{J}\}$ a *triangulation* with spatial step $h := \max_{k\in\mathcal{J}}(h_k)$ and interval I_k an *element*. By $\mathcal{E}_h$ we denote the smallest possible set of all endpoints of all subintervals I_k, i.e. $\mathcal{E}_h = \{x_0 = a, x_1, \ldots, x_{N-1}, x_N = b\}$. Further, we label by $\mathcal{E}_h^I$ the set of all inner nodes. Obviously, $\mathcal{E}_h = \mathcal{E}_h^I \cup \{a, b\}$.

DG method allows to treat with different polynomial degrees over elements. Therefore, we assign a positive integer p_k as a *local polynomial degree* to each $I_k \in \mathcal{T}_h$. Then we set the vector $\mathsf{p} = \{p_k, I_k \in \mathcal{T}_h\}$. Over the triangulation $\mathcal{T}_h$ we define the finite dimensional space of discontinuous piecewise polynomial functions

$$S_{h\mathsf{p}} \equiv S_{h\mathsf{p}}(I, \mathcal{T}_h) = \{v; v|_{I_k} \in P_{p_k}(I_k)\ \forall\, k \in \mathcal{J}\}, \tag{5}$$

where $P_{p_k}(I_k)$ denotes the space of all polynomials of degree $\leq p_k$ on I_k, $I_k \in \mathcal{T}_h$. Consequently, the approximate solution of the local problem (1)–(3) is sougth in the space $S_{h\mathsf{p}}$.

For each $x \in \mathcal{E}_h^I$ there exist two elements $I_k,\, I_{k+1} \in \mathcal{T}_h$ such that $I_k \cap I_{k+1} = \{x\}$. Let us denote

$$v(x^+) = \lim_{\varepsilon\to 0+} v(x+\varepsilon) \quad \text{and} \quad v(x^-) = \lim_{\varepsilon\to 0+} v(x-\varepsilon) \tag{6}$$

the *traces* of v at inner points of I. Moreover,

$$[v(x)] = v(x^-) - v(x^+), \qquad \langle v(x)\rangle = \frac{1}{2}\left(v(x^-) + v(x^+)\right), \tag{7}$$

denote the *jump* and *mean value* of function v at points $x \in \mathcal{E}_h^I$, respectively. By convention, we also extend the definition of jump and mean value for endpoints of domain I, i.e.

$$[v(x_0)] = -v(x_0^+), \quad \langle v(x_0)\rangle = v(x_0^+), \quad [v(x_N)] = v(x_N^-), \quad \langle v(x_N)\rangle = v(x_N^-) \tag{8}$$

Now, we recall the space semi-discrete DG scheme presented in [7]. The crucial item of the DG formulation of model problem is the treatment of the convective part. The convective terms are approximated with the aid of the following numerical flux $H(\cdot,\cdot)$ through node $x \in \mathcal{E}_h$ in the positive direction (i.e. outer normal is equal to one):

$$H\Big(u(x^-), u(x^+)\Big) = \begin{cases} c(x)\cdot u(x^-), & \text{if } c(x) > 0 \\ c(x)\cdot u(x^+), & \text{if } c(x) \leq 0 \end{cases}, \tag{9}$$

which is based on the concept of *upwinding*, see [5]. The choice of $u(x^-), u(x^+)$ for boundary points $\{a, b\}$ is necessary to specify. Here we use:

$$u(a^-) = u_D^a \quad \text{and} \quad u(b^+) = u_D^b. \tag{10}$$

A particular attention should be also paid to the treatment of the diffusive terms, in order to replace the inter-element continuity, we add some *stabilization* and *penalty* terms into formulation of the discrete problem.

Therefore, we say that $u_h \in C^1(0,T; S_{h\mathsf{p}})$ is the *semi-discrete solution* of the problem (1) – (3) if $(u_h(0), v_h) = (u^0, v_h)\ \forall\, v_h \in S_{h\mathsf{p}}$ and

$$\left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + b_h(u_h(t), v_h) + a_h^\Theta(u_h(t), v_h) + \nu_0 J_h^\sigma(u_h(t), v_h) = l_h^\Theta(v_h)\,(t) \tag{11}$$

$$\forall\, v_h \in S_{h\mathsf{p}},\ \forall\, t \in (0, T),$$

where $(\cdot,\cdot)$ denotes the $L^2(I)$-scalar product and

$$a_h^{\Theta}(u,v) = \sum_{k\in\mathcal{J}} \int_{I_k} \nu(x)\cdot\frac{\partial u(x,t)}{\partial x}\cdot v'(x)\,\mathrm{d}x - \sum_{x\in\mathcal{E}_h} \left\langle \nu(x)\cdot\frac{\partial u(x,t)}{\partial x}\right\rangle [v(x)] \\ +\Theta \sum_{x\in\mathcal{E}_h} \langle \nu(x)\cdot v'(x)\rangle\, [u(x,t)], \tag{12}$$

$$b_h(u,v) = -\sum_{k\in\mathcal{J}} \int_{I_k} c(x)\cdot u(x,t)\cdot v'(x)\,\mathrm{d}x + \sum_{x\in\mathcal{E}_h^I} H\Big(u(x^-,t),u(x^+,t)\Big)\,[v(x)] \\ -H\Big(u_D^a(t),u(x^+)\Big)\cdot v(a^+) \;+\; H\Big(u(x^-),u_D^b(t)\Big)\cdot v(b^-) \tag{13}$$

$$J_h^{\sigma}(u,v) = \sum_{x\in\mathcal{E}_h^I} \sigma(x)[u(x,t)]\,[v(x)] + \sigma(a)\cdot u(a^+,t)\cdot v(a^+) + \sigma(b)\cdot u(b^-,t)\cdot v(b^-) \tag{14}$$

$$l_h^{\Theta}(v)(t) = \int_I g(x,t)\cdot v(x)\,\mathrm{d}x - \Theta\nu(a)\cdot v'(a^+)\cdot u_D^a(t) + \Theta\nu(b)\cdot v'(b^-)\cdot u_D^b(t) \\ +\nu_0\sigma(a)\cdot u_D^a(t)\cdot v(a^+) + \nu_0\sigma(b)\cdot u_D^b(t)\cdot v(b^-). \tag{15}$$

According to value of parameter Θ, we speak of *symmetric* ($\Theta=-1$), *incomplete* ($\Theta=0$) or *nonsymmetric* ($\Theta=1$) variants of stabilization of DG method, i.e., we generally consider three variants of the diffusion form a_h^{Θ} and right-hand side form l_h^{Θ}. Penalty terms are represented by J_h^{σ} and the penalty parameter function $\sigma:\mathcal{E}_h\to I\!R$ in (14) and (15) is defined in spirit of [4] as

$$\sigma(x) = \frac{C_W}{d(x)} \quad \text{with} \quad d(x) = \begin{cases} h_1/{p_1}^2 & ,\ x=a, \\ \min\Big(h_k/p_k^2, h_{k+1}/p_{k+1}^2\Big) & ,\ x\in\mathcal{E}_h^I \wedge \{x\} = I_k\cap I_{k+1}, \\ h_N/{p_N}^2 & ,\ x=b, \end{cases} \tag{16}$$

where $C_W>0$ is a suitable constant depending on the used variant of scheme and on the degree of polynomial approximation.

The problem (11) represents a system of ordinary differential equations (ODEs) for $u_h(t)$ which has to be discretized in time by a suitable method. Since these ODEs belong to the class of stiff problems and due to linearity of convective and diffusive terms it is advantageous to use an *implicit* approach via the *backward Euler method.*

Let $0=t_0<t_1<\ldots<t_M=T$ be a partition of the interval $[0,T]$ and $\tau_l\equiv t_{l+1}-t_l$, $l=0,1,\ldots,M-1$. We define the *approximate solution* of problem (1)–(3) as functions $u_h^l\in S_{h\mathsf{p}}$, $u_h^l\approx u_h(t_l)$, $t_l\in[0,T]$, $l=1,\ldots,M$, satisfying the following identity

$$\frac{1}{\tau_l}\Big(u_h^{l+1}-u_h^l,v_h\Big) + a_h^{\Theta}\Big(u_h^{l+1},v_h\Big) + b_h\Big(u_h^{l+1},v_h\Big) \\ +\nu_0 J_h^{\sigma}\Big(u_h^{l+1},v_h\Big) = l_h^{\Theta}(v_h)\ (t_{l+1}) \qquad \forall\, v_h\in S_{h\mathsf{p}}, \tag{17}$$

where u_h^0 is $S_{h\mathsf{p}}$-approximation of u^0.

The resulting method is practically unconditionally stable, has a high order of accuracy with respect to the space coordinates and the first order of accuracy with respect to time. At each time instant $t_{k+1}\in[0,T]$, we have to solve only one system of linear algebraic equations representing by the discrete problem (17).

4 Numerical experiments

In this section, we consider three convection–diffusion problems in 1D with (piecewise) constant convection coefficient as well as viscosity. The whole algorithm is implemented in MATLAB and uses piecewise linear, quadratic and cubic approximations on partition of I with constant mesh size h and time step τ.

The first numerical example represents the case of continuous coefficients and steady-state solution. We set $c(x) = 1.0$, $\nu(x) = 0.1$, $I = (0, 1)$, $T = 1.0$ and define the function g and the initial and boundary conditions in such a way that the exact solution has the following form:

$$u(x,t) = 1 + 4(1 - e^{-10t})x(1 - x). \tag{18}$$

The mesh size $h = 0.01$ and the time step $\tau = 0.001$. We carried out computations by piecewise cubic approximations and set $\Theta = 0$ (incomplete variant). Figure 1 shows the comparison with exact solution (18) and development of approximation error $e_h = \|u_h^l - u(\cdot, t_l)\|_I$.

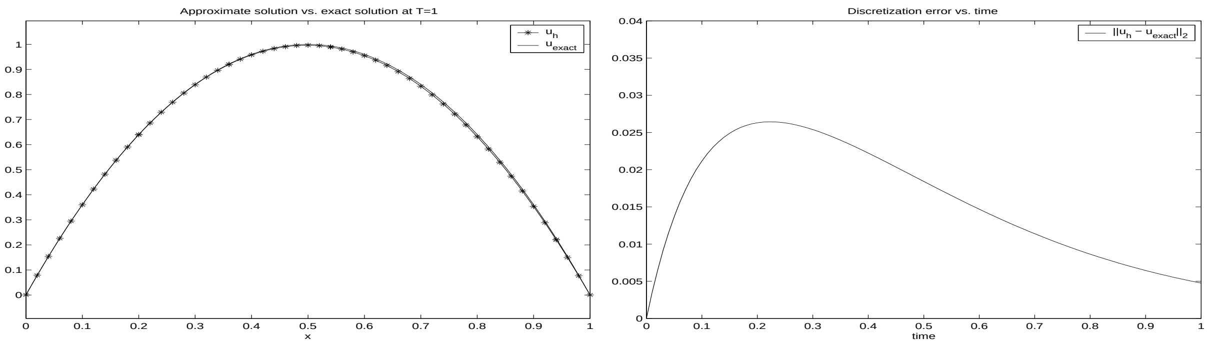

Figure 1: Continuous case (steady): Comparison of exact and approximate solution (left), development of discretization error (right)

In the second experiment, the parameters of computation were the same as in the first example except for function g and the initial and boundary conditions, which are chosen in such a way that the exact solution has the unsteady form:

$$u(x,t) = 1 + 4x(1 - x)t. \tag{19}$$

One can see in Figure 2 almost identical behavior of approximate solution in comparison with exact one.

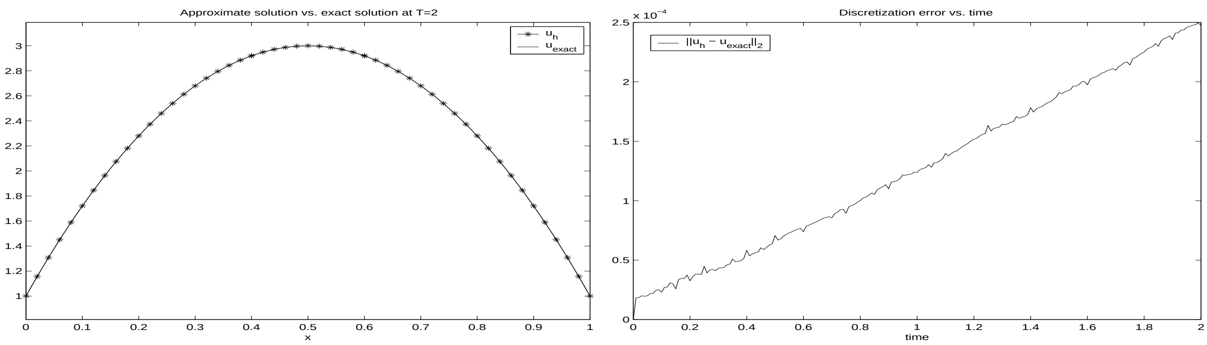

Figure 2: Continuous case (unsteady): Comparison of exact and approximate solution (left), development of discretization error (right)

The last experiment is devoted to the case of discontinuous piecewise constant convection and diffusion coefficients. Without loss of generality, we take

$$c(x) = \begin{cases} 0.1, & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 4.0, & \text{if } x \in \left(\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} 0.01, & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 1.0, & \text{if } x \in \left(\frac{1}{2}, 1\right] \end{cases}, \tag{20}$$

$g(x) \equiv 0$, $T = 0.6$ and the following initial and boundary conditions:

$$u(0,t) = u(1,t) = 1.0, \ t \in [0, T] \quad \text{and} \quad u(x, 0) = \exp\left(-1.2(1.2 - 6x)^2\right), \ x \in I. \tag{21}$$

The obtained satisfactory results, illustrating the potency of presented method, are depicted in Figure 3.

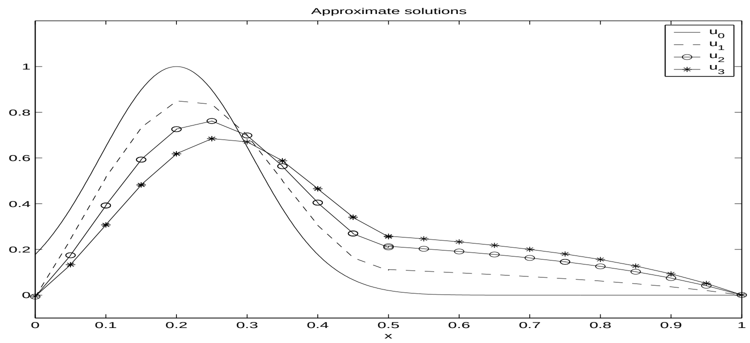

Figure 3: Discontinuous case: Development of approximate solutions, u_0 (initial condition), u_1 (solution at $t = 0.2$), u_2 (solution at $t = 0.4$) and u_3 (solution at $t = 0.6$)

5 Conclusion

We have dealt with the numerical solution of the linear convection-diffusion equation. We have presented DG approach together with the backward Euler method for spatial and time discretization, respectively. A set of numerical examples produced satisfactory results and illustrated the potency of the resulting scheme even for discontinuous convection coefficient and viscosity. For the future work, we intend to extend this method to parallel implementation consisting of domain decomposition techniques and a semi-implicit treatment of local subproblems with interface conditions.

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References

[1] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.

[2] B. Cockburn. Discontinuous Galerkin methods for convection dominated problems. In T. J. Barth and H. Deconinck, editors, *High–Order Methods for Computational Physics*, Lecture Notes in Computational Science and Engineering 9, pages 69–224. Springer, Berlin, 1999.

[3] B. Cockburn, G. E. Karniadakis, and C.-W. Shu, editors. *Discontinuous Galerkin Methods.* Springer, Berlin, 2000.

[4] V. Dolejší and J. Hozman. A priori error estimates for dgfem applied to non-stationary nonlinear convectiondiffusion equation. In G. Kreiss et. Al. Eds., editor, *Numerical Mathematics and Advanced Applications, ENUMATH 2009*, pages 459–468. Springer, 2010.

[5] M. Feistauer, J.Felcman, and I. Straškraba. *Mathematical and Computational Methods for Compressible Flow.* Oxford University Press, Oxford, 2003.

[6] A. Gilat. *MATLAB An Introduction with Applications.* John Wiley & Sons, Inc., 2004.

[7] B. Rivière. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation.* Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 2008.

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