A PROPOSAL OF ALGORITHM FOR SOLVING P-MEDIAN PROBLEM

V. Medvid'

Abstract

This paper deals with the weighted *p*-median problem of locating *p* facilities relative to a set of customers. Facilities *p* must be selected in such way that the sum of distances multiplied by the costs of vertices to the chosen vertices is minimal. This problem is a graph theory problem that was originally designed for, and has been extensively applied to, a facility location. We propose a new genetic algorithm for a well-known facility location problem. The algorithm is relatively simple and it generates good solutions quickly.

1 Introduction

This problem has a wide range of practical applications. A very well known and also a very important application is the optimal location of rescue services in the given territory, which can be represented by a state, a region, a county or a district.

The main criterion is the focus on a human being, which points out the necessity of the fastest possible service to the customer, as in many cases they have to save human lives.

Another very important criterion is the economic one, as it is important to save resources. This criterion is based on the fact that establishing a rescue service centre requires considerable financial resources.

Both above mentioned criteria are antagonistic, as the first one asks for establishing the highest possible number of service centres, where the other one pushes this number down in regard to high costs.

The given requirements can be summed up into one complex problem named "Optimal location of rescue service centres with respect to the needs of customers."

This problem is theoretically known as a p-median problem. Its task is stated as follows:

Let us have a planar graph which consists of evaluated vertices and evaluated edges. It is necessary to cover the greatest possible number of vertices *p*. Other vertices will be assigned to the covered vertices according to the shortest distances. Vertices *p* must be selected in such way that the sum of distances multiplied by the costs of vertices, to the chosen vertices is minimal.

This problem has also been solved by another techniques [1], [2], [3], [4], [5] and [6].

2 Model of the problem

There are given the following items:

- p is a number of centres placed,
- I is a set of customers,
- J is a set of candidates for the centre placements,
- c_i is a price of the i -th customer,
- d_{ij} is a distance from the i -th customer to the j -th center.

Decision variables

$x_j = 1$ if candidate j is used, 0 otherwise

$y_{ij} = 1$ if the requirement is fulfill by j -th center, 0 otherwise.

Objective function

$$min \sum_{i \in I} \sum_{j \in J} c_i d_{ij} y_{ij}$$

subject to

$$\sum_{j \in J} y_{ij} = 1$$

$$\sum_{j \in J} x_j = p$$

$y_{ij} \leq x_j, i \in I, j \in J$

$x_j \in \{0,1\}, j \in J$, $y_{ij} \in \{0,1\}, i \in I, j \in J$

The first condition means that each customer will be served by just one centre.The second condition implies p built service centres.The third condition ensures that the customer cannot be operated where the centre will not be built.The task is therefore to deploy some habitat service centres such that the sum of the distancesmultiplied by the costs of vertices from each customer to designated centres would be minimal.

3 Strategy of solution

The proposed algorithm solves this task by the following strategy. Firstly it selects p vertices with highest costs and assigns other vertices to the chosen ones according to the shortest distances. Then it chooses the vertex, which has the worst assignment regarding the product of its cost and the assigned distance. It becomes a new candidate for covering and the original vertex, to which it had been assigned moves temporarily to the backup. Then a new assignment is stated and compared with the original one. If this second assignment is better, the original is neglected. If it is worse, then the original assignment is kept. Then the algorithm selects another vertex with the worst assignment and the entire process is repeated until the set of vertices with the “worst” assignment is empty.

4 Algorithm of a weighted p-median problem

Arrange the candidates according to their cost in a decreasing order.

Select the first p candidates and state $C=\{z_{k_1}, z_{k_2}, \dots, z_{k_p}\}$.

Assign each candidate a customer; so that each customer is assigned to the closest candidate u_{k_i} (obviously some customers can be assigned to themselves if they themselves are candidates) and calculate the corresponding sums $s_{k_i} = c_{p_1} \cdot d_{p_1 k_i} + c_{p_2} \cdot d_{p_2 k_i} + \dots + c_{p_l} \cdot d_{p_l k_i}$, i=1,2,…,p.

Calculate $S_0 = s_{k_1} + s_{k_2} + \dots + s_{k_p}$. Put $S \coloneqq S_0$.

Construct a set Z consisting of all assigned customers to the candidates in a decreasing order according to the values of products $c_i \cdot d_{ij}$.

1. Put $r = 1$. From set Z select the customer z_i with the greatest value $c_i \cdot d_{ij}$ and proclaim him the temporary candidate u_i then leave the candidate u_j, to which the selected customer was assigned, temporarily in the backup and name him the customer z_j.

Make a new temporary assignment of customers to the candidates according to the closest distances, in the same way as at the beginning.

For each candidate u_{k_i} calculate $s_{k_i} = c_{p_1} \cdot d_{p_1 k_i} + c_{p_2} \cdot d_{p_2 k_i} + \cdots + c_{p_j} \cdot d_{p_j k_i}$, i=1,2,…,p.

Calculate $S_r = s_{k_1} + s_{k_2} + \cdots + s_{k_p}$.

If $S_r < S$, then put $S = S_r$, $C := (C - \{u_j\}) \cup \{u_i\}$, i.e. reject the original candidate u_j and state a new candidate u_i with a new assignment of customers.

If $S_r \geq S$, then keep the original candidate u_j, reject the candidate u_i, and also keep the original assignment of candidates. Put C:=C.

Put $Z := Z - \{z_i\}$, i.e. exclude the customer from the set Z, the set of potential candidates for centre placement.

$r = r + 1$.

If $Z \neq \emptyset$, go to 1, otherwise proclaim C as the optimal placement and S the optimal sum. The end.

5 Examples of a weighted *p*-median problem

This algorithm was tested on several examples. In this section are presented two selected examples.

Explanations for the follow figures:

The numbers in the circles represent the prices of the vertices, respectively, the population of the town or village. The numbers in connecting lines represent the edge evaluation, respectively, the distances between towns or villages. $z_i, i = 1,2,\ldots$ indicate the towns or villages.

Example 1

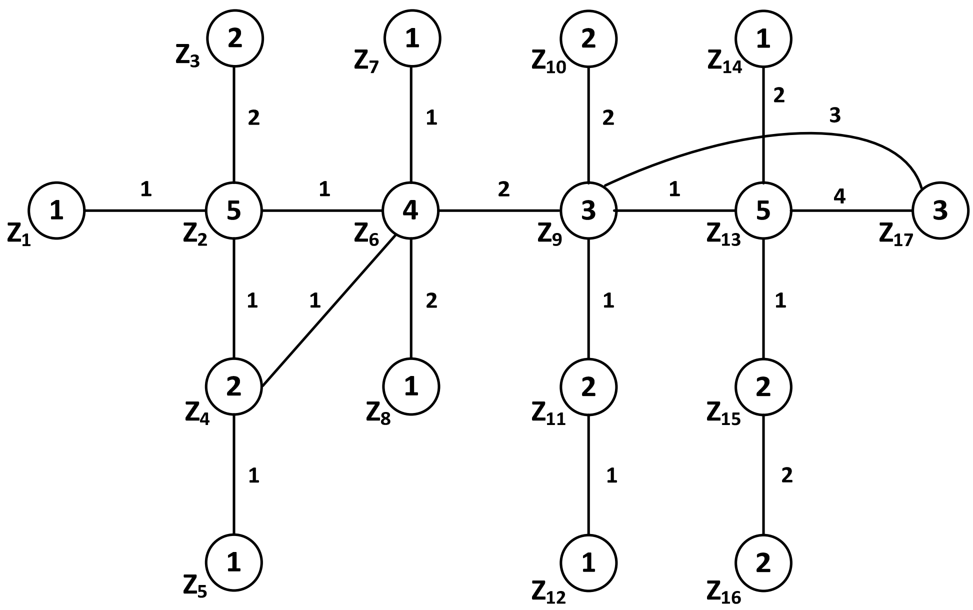

Figure 1

The graph in figure 1 consists of valued vertices and valued edges. This graph represents a network of towns and villages of a certain region, where the vertices represent the towns and villages, the values of vertices (= costs) represent the populations of the towns and villages; the edges with their values represent the distances between the towns or villages. Let the towns and villages be the customers, for whom it is necessary to place a security health service. Our task is to place the security health service centres to some of them. In our case we can afford placing only two centres, which means we are dealing with a 2 – median (*p*=2). These centres must be placed in such a way that the sum of individual products of costs and distances from customers the closest centres was the smallest.

Our algorithm will select two towns with largest populations and proclaim them the candidates for centre placement. In our case the two towns are z_2 and z_{13}, whose population is equal to 5. The other customer will be assigned to the candidates according to the shortest distances.

We have *p* =2.

$c(z_2) = c(z_{13}) = 5$.

$z_2 \leftarrow \{z_1, z_3, z_4, z_5, z_6, z_7, z_8\}$,

$z_{13} \leftarrow \{z_9, z_{10}, z_{11}, z_{12}, z_{14}, z_{15}, z_{16}, z_{17}\}$.

s_2 =1.1+2.2+2.1+1.2+4.1+1.2+1.3=18.

$s_{13} = 3.1 + 2.3 + 2.2 + 1.3 + 1.2 + 2.1 + 2.3 + 3.4 = 38$.

$S_0 = s_2 + s_{13} = 56,$

$S := S_0$.

Create the initial set of candidates $C = \{z_2, z_{13}\}$.

Now the algorithm will create a list of other customers, who can also become the candidates for centre placement. This list will not include the customers, who stand at the end and their cost is smaller or equal to their closest neighbour, because they are not prospective to become candidates. So $Z = \{z_4, z_6, z_9, z_{11}, z_{13}, z_{17}\}.$

Put the counter of candidates $r = 1$. Then it selects a customer with the greatest value of $c_i . d_{ij}$ from the list of customers. It states him a candidate and the original candidate, to which the selected customer was assigned, is placed in the backup. In this case it is the customer z_{17}. This customer was assigned to the customer z_{13}, which means that z_{13} is placed in the backup. Now the algorithm states the new assignment of customers to the candidates according to the shortest distances. In this case the assignment of candidate z_2 does not change and the customers z_{17} and z_{13} will change their positions, i.e. $z_2 \leftarrow \{z_1, z_3, z_4, z_5, z_6, z_7, z_8\}$ a $z_{17} \leftarrow \{z_9, z_{10}, z_{11}, z_{12}, z_{13}, z_{14}, z_{15}, z_{16}\}$. The sum s_2 stays unchanged and $s_{17} = 3.3 + 2.5 + 2.4 + 1.5 + 5.4 + 1.6 + 2.5 + 2.7 = 82$.

$S_1 = s_2 + s_{17} = 100$. $S = 56, S_1 > S$ that is why the algorithm rejects the new candidate z_{17} and keeps the original candidate z_{13}. The customer z_{17} will be excluded from the list of customers, i.e. $Z = Z - \{z_{17}\}, Z = \{z_4, z_6, z_9, z_{11}, z_{13}\}.$

$Z \neq \emptyset$, so the counter will increase the number of customers into $r = r + 1$ and selects another customer from the list *Z* with the greatest value $c_i . d_{ij}$ to become the candidate for centre placement.

It will be the customer z_6 with a value 4.1=4, where 4 is its cost and 1 its distance from the customer z_2. In the same way as it was done previously, the algorithm will create a new assignment. Now the assignment to z_{13} stays unchanged and z_2 and z_6 will change their positions, i.e. $z_6 \leftarrow \{z_1, z_2, z_3, z_4, z_5, z_7, z_8\}$. Their corresponding sums $s_6 = 1.2 + 5.1 + 2.3 + 2.1 + 1.2 + 1.1 + 1.2 = 20,$ $s_{13} = 38$.

$S_2 = s_6 + s_{13} = 58, S_2 > S$. So the new candidate z_6 is rejected and the original candidate z_2 is kept. The customer z_6 is excluded from the list *Z*, i.e. $Z = Z - \{z_6\}$, $Z = \{z_4, z_9, z_{11}, z_{15}\}$.

$Z \neq \emptyset$, so the counter will increase the number of customers, i.e. $r = r + 1$. In a similar way as it was done previously, the algorithm will select another candidate z_{11}. The algorithm will create a new assignment. The candidate z_2 will stay with the original assignment and candidate z_{11} will change its position with customer z_{13}. So $z_{11} \leftarrow \{z_9, z_{10}, z_{12}, z_{13}, z_{14}, z_{15}, z_{16}, z_{17}\}$, the corresponding sums are s_2=1.1+2.2+2.1+1.2+4.1+1.2+1.3=18 and $s_{11} = 3.1 + 2.3 + 1.1 + 5.2 + 1.4 + 2.3 + 2.5 + 3.4 = 52$.

$S_3 = s_2 + s_{11} = 70, S_3 > S$.

It rejects the new candidate z_{11} and keeps the original candidate z_{13}. At the same time it excludes z_{11} from the list of customers *Z*, which means.

$Z \neq \emptyset$, so the counter will increase the number by 1, so $r = r + 1$. Then the algorithm selects a new customer z_9 according to the above mentioned criterion and creates a new assignment. The assignment of customers to the candidate z_2 stays unchanged and, as it was done before, the new candidate z_9 will change its position with the original candidate z_{13}. So $z_9 \leftarrow \{z_{10}, z_{12}, z_{13}, z_{14}, z_{15}, z_{16}, z_{17}\}$. The sum s_2=1.1+2.2+2.1+1.2+4.1+1.2+1.3=18 and the sum $s_9 = 2.2 + 2.1 + 1.2 + 5.1 + 1.3 + 2.2 + 2.4 + 3.3 = 37$.

$S_4 = s_2 + s_9 = 55, S = 56, S_4 < S$, so it states a new candidate z_9 and rejects candidate z_{13}. At the same time it makes changes in the set of candidates, it means $C = \{z_2, z_9\}$. Also the S will have a smaller assigned value 55 from this step, i.e. $S := S_4$. Obviously the customer z_9 will not be in the set Z. So $Z = \{z_4, z_{15}\}$.

$Z \neq \emptyset$, which means it continues in selection of another customer from this set to become the candidate for centre placement.

Both customers z_4 and z_{15} have equal values 2.1=2, so the algorithm will select the first one in order, which is the customer z_4. Candidate z_2 is temporarily moved to backup and a new assignment is created so that z_4 and z_2 change their positions, i.e. $z_4 \leftarrow \{z_1, z_2, z_3, z_5, z_6, z_7, z_8\}$ and $z_9 \leftarrow \{z_{10}, z_{12}, z_{13}, z_{14}, z_{15}, z_{16}, z_{17}\}$, so that the assignment to z_9 stays unchanged. Now the corresponding sums are $s_4 = 1.2 + 5.1 + 2.3 + 1.1 + 4.1 + 1.2 + 1.3 = 23$ and $s_9 = 2.2 + 2.1 + 1.2 + 5.1 + 1.3 + 2.2 + 2.4 + 3.3 = 37$.

$S_5 = s_4 + s_9 = 60, S = 55, S_5 > S$, so the new candidate is rejected and the original candidate z_2 is kept. The customer z_4 is excluded from the list. So that $Z = \{z_{15}\}$.

Finally, the last customer z_{15} is selected to become the candidate and candidate z_9 is temporarily moved to the backup. The assignment for the candidate z_2 stays unchanged and in a similar way as it was done before, candidates z_{15} and z_9 will change their position, so $z_{15} \leftarrow \{z_9, z_{10}, z_{12}, z_{13}, z_{14}, z_{16}, z_{17}\}$ and $z_2 \leftarrow \{z_1, z_3, z_4, z_5, z_6, z_7, z_8\}$.

The corresponding sums are as follows $s_{15} = 3.2 + 2.4 + 1.4 + 5.1 + 1.3 + 2.2 + 3.4 = 48$ and s_2=1.1+2.2+2.1+1.2+4.1+1.2+1.3=18.

$S_6 = s_2 + s_{15} = 76, S = 55, S_6 > S$, so the new candidate z_{15} is rejected and the original candidate z_9 is kept. The customer z_{15} is excluded from the list, so $Z = \emptyset$.

The algorithm will finish the procedure and the final assignment of candidates for centre placement is $C = \{z_2, z_9\}$ with the final minimal sum $S = 55$, where $z_2 \leftarrow \{z_1, z_3, z_4, z_5, z_6, z_7, z_8\}$ and $z_9 \leftarrow \{z_{10}, z_{12}, z_{13}, z_{14}, z_{15}, z_{16}, z_{17}\}$.

Example 2

It isgiven the networkof towns and villages(see figure 2). Similarly as in the previous example, the problem is to find three candidates, i. e. *p*=3. Theprocedureisthe sameasin the previous example, therefore itwill not bedescribed in detail. We give only the final solution.

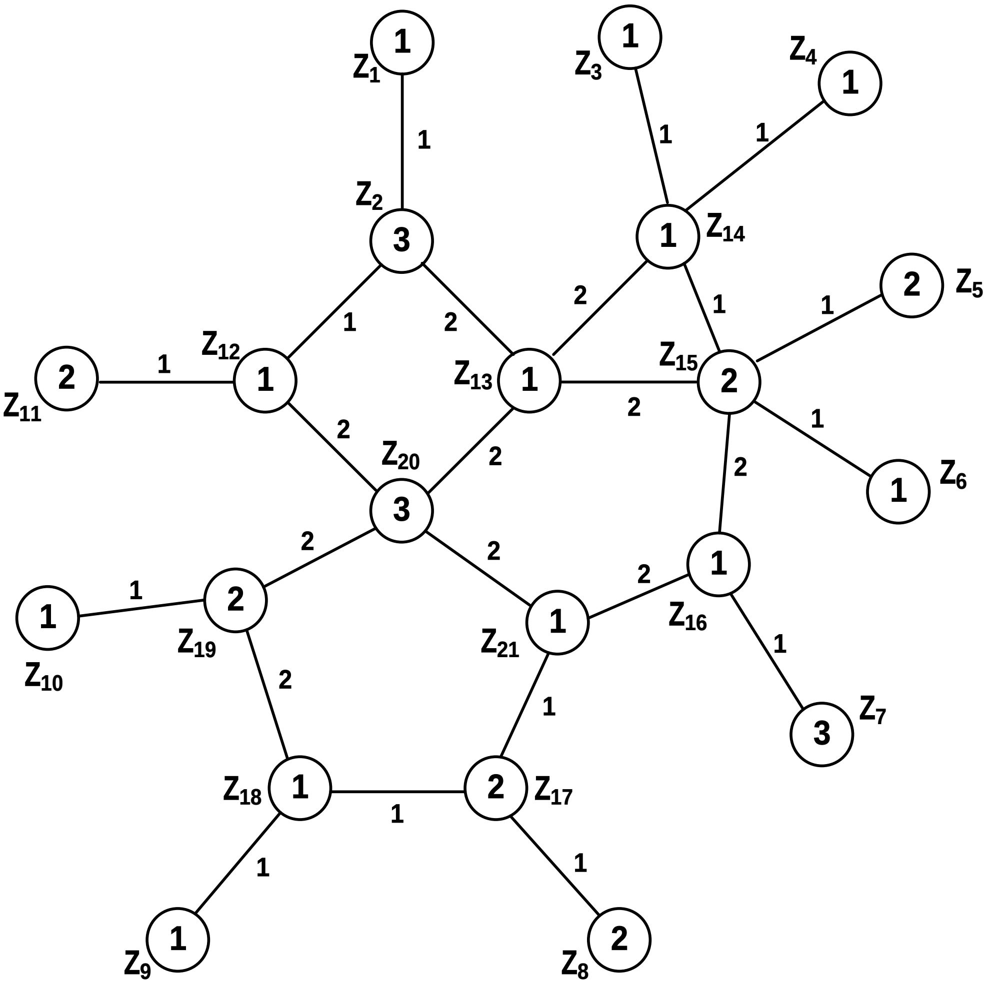

Figure 2

The solution is as follows:

$C = \{z_2, z_{15}, z_{18}\}$,

$z_2 \leftarrow \{z_1, z_{11}, z_{12}, z_{13}\}$,

$z_{15} \leftarrow \{z_3, z_4, z_5, z_6, z_{14}, z_{16}\}$,

$z_{18} \leftarrow \{z_8, z_9, z_{10}, z_{17}, z_{19}, z_{20}, z_{21}\}$,

$S = 52$.

6 Conclusion

Author carried out the implementation of the algorithm on several different types of graph structures using Matlab environment. Algorithm always found the optimal solution.

Notwithstanding the above, it should be noted that the algorithm must be implemented by programming means for specific datasets.

Consequently, it is necessary to compare the results with the exact solutions to smaller ranges of inputs such that the exact algorithms are not able to solve in the "reasonable" time.

Acknowledgement

This work has been supported by the research grant KEGA 088 zU-4/2011.

References

[1] GÁBRIŠOVÁ, L., JANÁČEK, J.:Návrhrozmiestneniastaníczáchrannejslužby v Žilinskomkraji. InSborníkpříspěvku, Úlohydiskrétníoptimalizace v dopravnípraxi 2013, ISBN 978-80-7395-662-2, pp. 16-25, Pardubice 2013

[2] JANÁČKOVÁ, M., SZENDREYOVÁ, A.: Vplyvrozdeleniadĺžokv maticivzdialenostínaefektívnosťexaktnéhoalgoritmu v umiestňovacejúlohe s obmedzenýmpočtomstredísk. In Sborníkpříspěvku, Úlohydiskrétníoptimalizace v dopravnípraxi 2013, ISBN 978-80-7395-662-2, pp. 59-65, Pardubice 2013

[3] JÁNOŠIKOVÁ, Ľ., MÁRTON, P., ŽARNAY, M.: Overovanievýsledkovmatematickýchmodelovverejnýchobslužnýchsystémovpomocoupočítačovejsimulácie – výbersimulačnéhonástroja, In Sborníkpříspěvku, Úlohydiskrétníoptimalizace v dopravnípraxi 2013, ISBN 978-80-7395-662-2, pp. 66-74, Pardubice 2013

[4] KVET, M.: Pokrývaciemetódynávrhuštruktúrverejnýchobslužnýchsystémov, In Sborníkpříspěvku, Úlohydiskrétníoptimalizace v dopravnípraxi 2013, ISBN 978-80-7395-662-2, pp. 75-90, Pardubice 2013

MEDVIĎ, V.: Návrhoptimálnehoalgoritmu pre úlohu*p*- medián, In Sborníkpříspěvku, Úlohydiskrétníoptimalizace v dopravnípraxi 2013, ISBN 978-80-7395-662-2, pp. 91-96, Pardubice 2013

PEŠKO, Š.: Metóda čiastočných súčtov pre p – medián v dopravnýchsieťach, In Sborníkpříspěvku, Úlohydiskrétníoptimalizace v dopravnípraxi 2013, ISBN 978-80-7395-662-2, pp. 104-111, Pardubice 2013

VladimírMedviď

Department of Mathematics, Faculty of Humanities, University of Zilina, Slovakia,

E-mail: vladimir.medvid@fhv.uniza.sk