# WAVELET TRANSFORM
# IN SIGNAL AND IMAGE DE-NOISING

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**Abstract**

**Signal and image transforms represent an efficient tool for observed data analysis and further processing. The paper is devoted to the description of selected properties of Wavelet transform based upon the application of dilated and translated time limited functions enabling multiresolution signal analysis.**

## 1 Introduction

Wavelet transform represents a mathematical tool for one-dimensional or multi-dimensional signal analysis and processing. The paper is devoted to the study of analytical description of Wavelet functions and their definition by solution of dilation equations. Conditions of signal perfect reconstruction are summarized in this connection to define dilation equation coefficients.

The main part of the paper presents algorithms for signal and image decomposition enabling the following signal de-noising using selected thresholding methods and similar approach to one-dimensional and two-dimensional signal processing.

## 2 Principles of Signal Wavelet Decomposition

Signal Wavelet decomposition using Wavelet transform (WT) provide an alternative to the short-time Fourier transform (STFT) for signal analysis [2, 1] resulting in signal decomposition into two-dimensional function of time and scale.

Wavelet functions used for signal analysis are derived from the initial function $W(t)$ forming basis for the set of functions

$$W_{m,k}(t) = \frac{1}{\sqrt{a}}\, W\,(\frac{1}{a}\,(t-b)) = \frac{1}{\sqrt{2^m}}\, W\,(2^{-m}t - k))$$

for discrete parameters of dilation $a = 2^m$ and translation $b = k\ 2^m$. Wavelet dilation closely related to its spectrum compression enables local and global signal analysis. An example of an analytically defined Wavelet function is presented in Fig. 1.

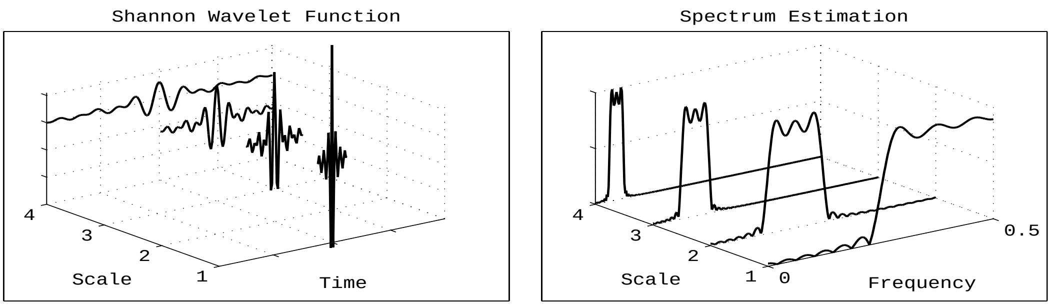


Figure 1: Shanon Wavelet function derived from the initial function defined in the form of relation $W(t)$=sin$(\pi t/2)$ cos$(3\pi t/2)/(\pi t/2)$ and the effect of its dilation to spectrum compression

### 2.1 Definition of Wavelet Function

Wavelet functions can be defined either in the analytical form or by solution of dilation equations. In case of Daubechies Wavelet functions [1] with four coefficients the initial Wavelet function is defined by solution of equation

$$W(x) = -c_3\phi(2x) + c_2\phi(2x-1) - c_1\phi(2x-2) + c_0\phi(2x-3)$$

where $\Phi(x)$ is a Scaling function with the same coefficients. These coefficients are chosen thoroughly in order to generate good Wavelets with special properties. The basic dilatation equation has a form

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + c_2\phi(2x-2) + c_3\phi(2x-3)$$

Solution of this dilation equation can be obtained by an iterative algorithm evaluating

$$\phi_j(x)=c_0\phi_{j-1}(2x) + c_1\phi_{j-1}(2x-1) + c_2\phi_{j-1}(2x-2) + c_3\phi_{j-1}(2x-3)$$

for $j = 1, 2, \cdots$ until $\phi_j(x)$ becomes indistinguishable from $\phi_{j-1}(x)$. It is possible to start from a box function $\phi_0(x) = 1$, $0 < x \leq 1$, $\phi_0(x) = 0$ elsewhere. Scaling and Wavelet functions must satisfy some conditions to enable perfect signal reconstruction - conservation of area condition, accuracy condition and the orthogonality condition. The area under the Scaling function is conserved if the sum of the coefficients is equal to 2 forming relation

$$c_0 + c_1 + c_2 + c_3 = 2$$

Since a unit area box is used to start the iteration, the area under the Scaling function remains equal to unity expressed by relation $\int_{-\infty}^{\infty} \Phi(x)\, dx = 1$. The next condition for the Wavelet coefficients for a faithful representation of the analysing signal is that the Fourier transform of the Scaling function must be periodically zeros of the highest possible order. That will be guaranteed by conditions

$$c_0 - c_1 + c_2 - c_3 = 0 \quad \text{and} \quad -c_1 + 2c_2 - 3c_3 = 0$$

The development of the Scaling function $\phi(x)$ by iteration from a unite box must be reversible. From this claim we have the condition

$$c_0c_2 + c_1c_3 = 0$$

Conditions described above have resulted in four equations for four coefficients of dilatation equation with their solution

$$c_1 = (1 + \sqrt{(3)})/4 \quad c_2 = (3 + \sqrt{(3)})/4 \quad c_3 = (3 - \sqrt{(3)})/4 \quad c_4 = (1 - \sqrt{(3)})/4$$

Five steps of iteration process using algorithm presented in Fig. 2 are illustrated in Fig. 3.

```
  function [S,W]=dw(M)
% Scaling and Wavelet Function Iterative Estimation Using Dilation Equations
  global c; d=0.01; x=0:d:3;
  c(1)=(1+sqrt(3))/4; c(2)=(3+sqrt(3))/4; c(3)=(3-sqrt(3))/4; c(4)=(1-sqrt(3))/4;
  for k=1:M
    S=daub(x,k);
    W=-c(4)*daub(2*x,k)+c(3)*daub(2*x-1,k)-c(2)*daub(2*x-2,k)+c(1)*daub(2*x-3,k);
    figure(1); subplot(M,2,2*k-1); plot(x,S); axis([-inf +inf -2 2]); grid
                 if k==1, title('SCALING FUNCTION'), end
               subplot(M,2,2*k); plot(x,W); axis([-inf +inf -2 2]); grid
                 if k==1, title('WAVELET FUNCTION'), end
  end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  function S=daub(x,j)
% Scaling Function Definition
  global c
  if j==1
   S=x>0 & x<=1;
  else
   S=c(1)*daub(2*x,j-1)+c(2)*daub(2*x-1,j-1)+...
   c(3)*daub(2*x-2,j-1)+c(4)*daub(2*x-3,j-1);
  end
```

Figure 2: Algorithm of iterative solution of dilation equations

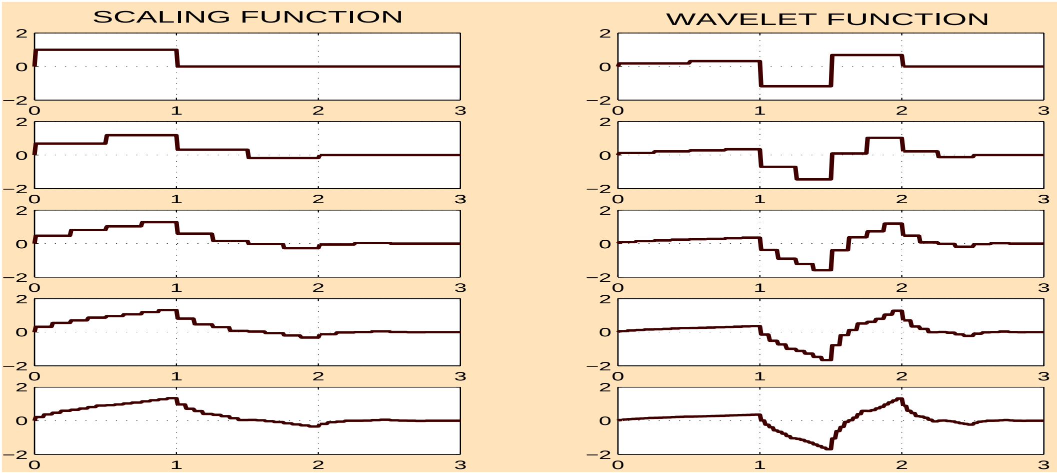


Figure 3: Recurrent solution of dilatation equations

### 2.2 Wavelet Shape Estimation Using Upsampling

Having a sequence $\{x_n\}_{n=0}^{N-1}$ $(T_s = 1)$ it is possible to define corresponding diluted sequence $\{z_n\}_{n=0}^{2N-1} = \{x_0\ 0\ x_1\ 0\ldots\}$ $(T_s = 0.5)$. After the elimination of fast signal components from diluted sequence we obtain a sequence with original values of the sequence $\{x_n\}_{n=0}^{N-1}$ interleaved by interpolated values replacing inserted zeros. We apply this process to the sequence of Wavelet coefficients $h_n = \{-c_3, c_2, -c_1, c_0\}$. Multiplication in the frequency domain corresponds to a convolution in the time domain. The convolution of a diluted sequence $z_n = \{-c_3,\ 0,\ c_2,\ 0,\ -c_1,\ 0,\ c_0\}$ and the sequence of the Scaling coefficients $s_k = \{c_0,\ c_1,\ c_2,\ c_3\}$ forms the next step of the iterative process of signal convolution according to relation $s_n \star z_n = \sum_{k=0}^{3} s_k\ x_{n-k}$. The algorithm in Fig. 4 provides functions in Fig. 5.

```
% Wavelet and Scaling Function Evaluation
  delete(get(0,'children'))
% Low-pass Daubechies Wavelet Filter Definition
  k=menu('Daubechies Wavelet Coefficients Definition',
         'Using Table from Wavelet Toolbox',...
         'Evaluation from Definition Equations for L=4');
  if k==1, load db2; l=db2*sqrt(2);
  else
     [l0,l1,l2,l3]=solve('l0+l1+l2+l3=2','l0-l1+l2-l3=0',...
                         '-l1+2*l2-3*l3=0','l0*l2+l1*l3=0');
     lsym=[l0(2),l1(2),l2(2),l3(2)]; l=numeric(lsym)/sqrt(2);
  end
% Complementary High-Pass Filter Definition
  h=fliplr(l); ll=length(l);for i=1:2:ll, h(i)=-h(i); end
% Daubechies Wavelet Function Shape Evaluation for N=4
% Using Basic Coefficients Upsampling and Convolution
  l0=l;
  for i=1:10
    hh=dyadup(h,2); h=conv(l0,hh); N=length(h); h=sqrt(2)*h;
    subplot(2,1,1); plot([1:N]/N,h); title('WAVELET FUNCTION')
  end
% Daubechies Scaling Function Shape Evaluation for N=4
% Using Basic Coefficients Upsampling and Convolution
  for i=1:10
    hh=dyadup(l,2); l=conv(l0,hh); N=length(l); l=sqrt(2)*l;
    subplot(2,1,2); plot([1:N]/N,l); title('SCALING FUNCTION')
  end
```

Figure 4: Algorithm for Daubechies coefficients evaluation and Wavelet shape evaluation using upsampling and convolution

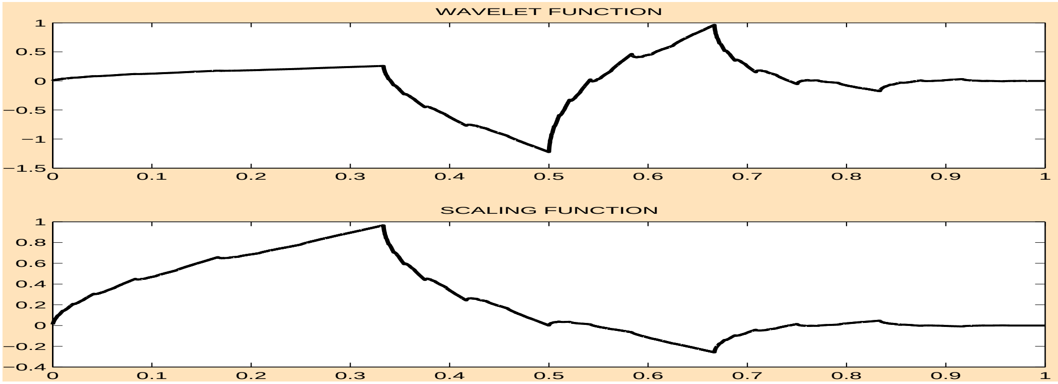


Figure 5: Convolution and Wavelet definition

### 2.3 Signal Wavelet Decomposition

Wavelet transform coefficients can be evaluated using Mallat decomposition tree presented in Fig. 6. This decomposition assumes in each step convolution of a signal with high-pass half-band filter (*Wavelet function*) and complementary low-pass filter (*Scaling function*). The high-pass signal part contains the finest details while the low-pass signal part contains slowly changing signal components. Signal downsampling is then applied in each decomposition step.

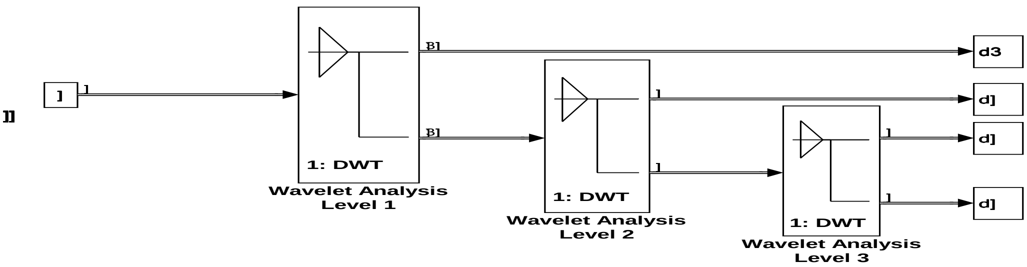


Figure 6: Signal decomposition by Mallat decomposition tree into three levels

An example of simulated signal decomposition is presented in Fig. 7. Discrete Wavelet transform of a signal $s1$ into levels $L = 3$ by Daubechies Wavelet function $db2$ using command

```
L=3; [c,l] = wavedec(s1,1,'db2');
```

provides coefficients **c** that can be ordered in one row vector enabling the following signal de-noising and reconstruction.

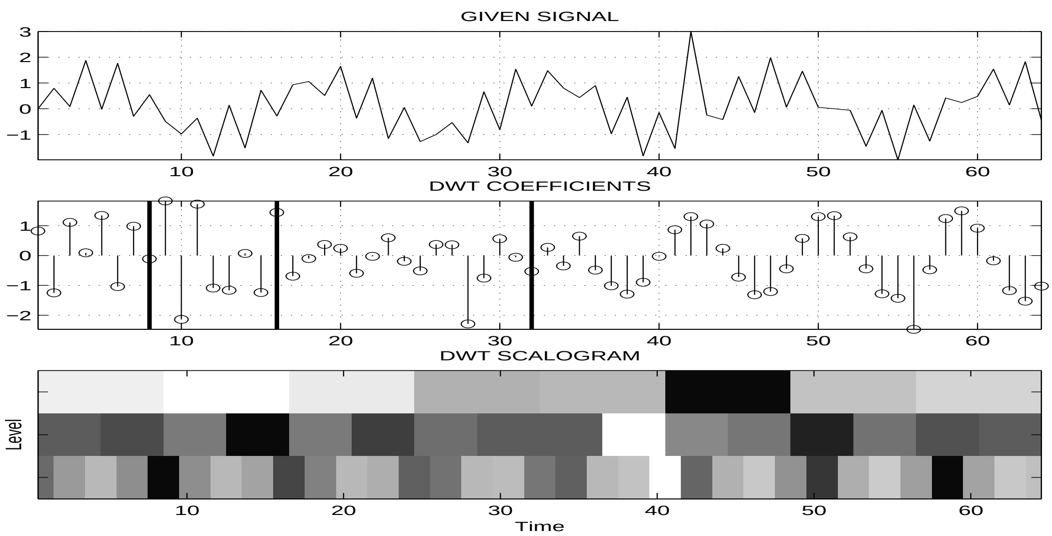


Figure 7: Simulated signal Wavelet decomposition into three levels presenting corresponding Wavelet coefficients pointing to signal impulse component

# 3 Image Wavelet Decomposition

Image Wavelet decomposition presented in Fig. 8 uses the same principles as that of signal decomposition. The only difference is in the fact that each column of image matrix is convolved with high-pass and low-pass filter followed by downsampling at first and then the same process is applied to image matrix rows. Each step of image decomposition results in four image matrices of the number of rows and columns reduced to the half of that of the original matrix.

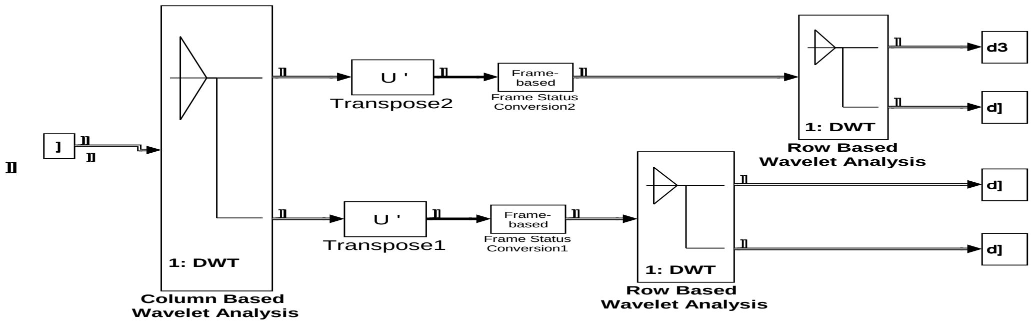


Figure 8: Image Decomposition

Results of one step image decomposition for a simulated image is presented in Fig. 9. Discrete Wavelet transform of an image $s2$ into the level $L = 1$ by function $db2$ using command

```
[c,l]=wavedec2(s2,1,'db2');
```

provides coefficients $\mathbf{c}$ that can be ordered in one row vector forming similar vector as that obtained in the one-dimensional case.

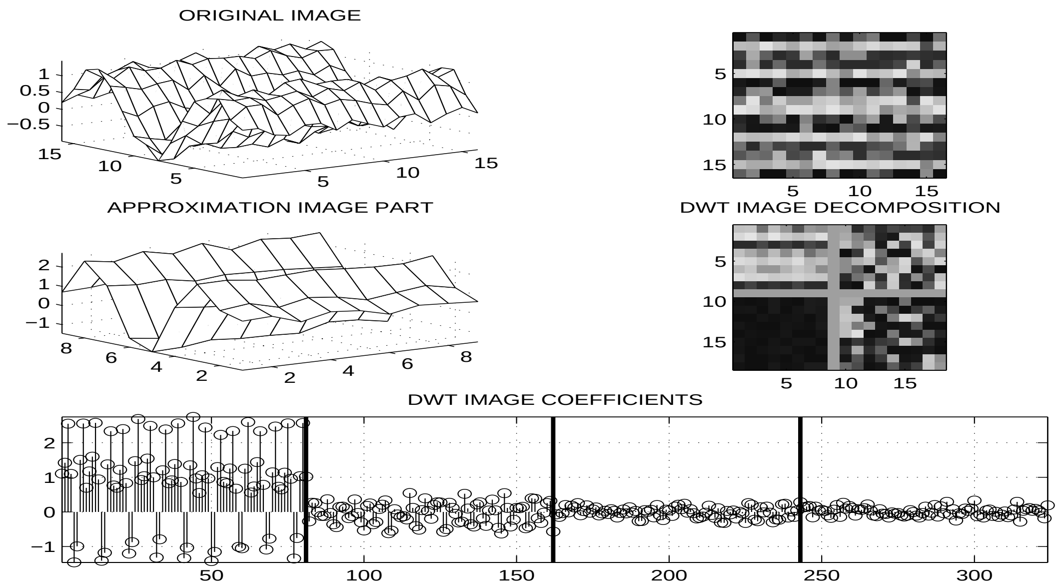


Figure 9: Simulated image one step Wavelet decomposition

# 4 Signal and Image De-Noising

Both in the case of one-dimensional and two-dimensional signal decomposition it is possible to modify resulting coefficients $\mathbf{c}$ before the following signal reconstruction to eliminate undesirable signal components. Methods of such a process assume estimation of appropriate threshold limits studied in various books and papers [3].

In case of soft thresholding it is possible to evaluate new coefficients $\overline{c}(k)$ using original coefficients $c(k)$ for a chosen threshold limit $\delta$ using algorithm presented in Fig. 10 by relation

$$\overline{c}(k) = \begin{cases} \text{sign } c(k)\ (|c(k)| - \delta) & \text{if } |c(k)| > \delta \\ 0 & \text{if } |c(k)| \le \delta \end{cases}$$

```
 function rs=recon4(s,w,level,lambda);
 if ~exist('s')
% Input Values Definition
   N=256; n=0:N-1; f1=0.02; f2=0.15;      % 1. Given signal
   s=sin(2*pi*f1*n)+sin(2*pi*f2*n);s(150)=s(150)+0.5;s=s-mean(s);
   w='db2';    % 2. Choise of wavelet function
   level=4;    % 3. Decomposition level
   lambda=2;   % 4. Threshold coefficient lambda
 end
% Signal Decomposition to a Given Level
    [c,l]=wavedec(s,level,w);
% Signal Reconstruction at a Given Level
    a=wrcoef('a',c,l,w,level); d=wrcoef('d',c,l,w,level);
% Signal Reconstruction
    i=find(abs(c)<=lambda); cd(i)=0;
    j=find(abs(c)>lambda); cd(j)=sign(c(j)).*(abs(c(j))-lambda);  z=waverec(cd,l,w);
% Plots
   subplot(5,1,1); plot(s,'r'); grid on; axis tight; v=axis;
     title(['GIVEN SIGNAL DECOMPOSTION AT LEVEL ',num2str(level)])
   subplot(5,1,2);plot(a);grid on;set(gca,'XtickLabel',[]);axis(v)
   subplot(5,1,3);plot(d);grid on;set(gca,'XtickLabel',[]);axis(v)
   subplot(5,1,4); if length(c)<=300, bar([c' cd'],'group'),
                    else, plot([c' cd']); end; grid on
   set(gca,'XtickLabel',[]); axis tight; v=axis; ll=length(l);
    for i=2:length(l)-1
      line([l(i);l(i)],[v(3);v(4)])
    end
     line([v(1);v(2)],[lambda;lambda]);
     line([v(1);v(2)],[-lambda;-lambda])
     title('SIGNAL WAVELET COEFFICIENTS')
   subplot(5,1,5); plot(z,'r'); grid on; axis tight; v=axis;
     title('SIGNAL RECONSTRUCTION')
```

Figure 10: Algorithm for signal de-noising using Wavelet decomposition and soft thresholding

## Acknowledgments

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## References

[1] D. E. Newland. *An Introduction to Random Vibrations, Spectral and Wavelet Analysis.* Longman Scientific & Technical, Essex, U.K., third edition, 1994.

[2] G. Strang. Wavelets and Dilation Equations: A brief introduction. *SIAM Review*, 31(4):614–627, December 1989.

[3] G. Strang and T. Nguyen. *Wavelets and Filter Banks.* Wellesley-Cambridge Press, 1996.

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