COMPLEX WAVELET TRANSFORM IN SIGNAL AND IMAGE ANALYSIS

MUSOKO VICTOR, PROCHÁZKA ALEŠ
Institute of Chemical Technology, Department of Computing and Control Engineering
Technická 1905, 166 28 Prague 6, Czech Republic
Phone: 00420-2-2435 4198 ∗ Fax: 00420-2-2435 5053
E-mail: Victor.Musoko@vscht.cz, A.Procházka@ieee.org

Abstract: Wavelet techniques can be successfully applied in various signal and image processing methods, namely in image denoising, segmentation, classification and motion estimation. The paper discusses the application of complex discrete wavelet transform (CDWT) which has significant advantages over real wavelet transform for certain signal processing problems. CDWT is a form of discrete wavelet transform, which generates complex coefficients by using a dual tree of wavelet filters to obtain their real and imaginary parts. What makes the complex wavelet basis exceptionally useful for denoising purposes is that it provides a high degree of shift-invariance and better directionality compared to the real DWT. The main part of the paper is devoted to the theoretical analysis of complex wavelet transform and its verification for simulated images. Resulting algorithms are then applied to the analysis and denoising of magnetic resonance biomedical images.

Keywords: Complex Discrete Wavelet Transform (CDWT), Dual-Tree, Filter Bank, Shift Invariance, Optimal Thresholding

1 INTRODUCTION

Complex wavelets have not been used widely in image processing due to the difficulty in designing *complex* filters which satisfy a perfect reconstruction property. To overcome this Kingsbury [2] proposed a *dual-tree* implementation of the CWT (DT CWT) which uses two trees of real filters to generate the real and imaginary parts of the wavelet coefficients separately. The two trees are shown in Fig. 3 for 1D signals. Even though the outputs of each tree are downsampled by summing the outputs of the two trees during reconstruction we are able to suppress the aliased components of the signal and achieve approximate shift invariance. In this paper CDWT which is an alternative to the basic DWT will be discussed, the DWT suffers from the following two problems:

- Lack of shift invariance - this results from the downsampling operation at each level. When the input signal is shifted slightly, the amplitude of the wavelet coefficients varies so much.
- Lack of directional selectivity - as the DWT filters are real and separable the DWT cannot distinguish between the opposing diagonal directions.

These problems hinder the use of wavelets in other areas of image processing. The first problem can be avoided if the filter outputs from each level are not downsampled but this increases the computational costs significantly and the resulting undecimated wavelet transform still cannot distinguish between opposing diagonals since the transform is still separable. To distinguish opposing diagonals with separable filters the filter frequency responses are required to be asymmetric for positive and negative frequencies. A good way to achieve this is to use complex wavelet filters which can be made to suppress negative frequency components. As we shall see the CDWT has improved shift-invariance and directional selectivity than the separable DWT.

2 PRINCIPLE OF CDWT AND THE DUAL TREE IMPLEMENTATION

The dual-tree CWT comprises of two parallel wavelet filter bank trees that contain carefully designed filters of different delays that minimize the aliasing effects due to downsampling [3]. The dual-tree CDWT of a signal $x(n)$ is implemented using two critically-sampled DWTs in parallel on the same data, as shown in Fig. 3. The transform is two times expansive because for an N-point signal it gives $2N$ DWT coefficients. If the filters in the upper and lower DWTs are the same, then no advantage is gained. So the filters are designed in a specific way such that the subband signals of the upper DWT can be interpreted as the real part of a complex wavelet transform and subband signals of the lower DWT can be interpreted as the imaginary part. When designed in this way the DT CDWT is nearly shift invariant, in contrast to the classic DWT.

2.1 Translation Invariance by Parallel Filter Banks

The orthogonal [6] two-channel filter banks with analysis low-pass filter given by the z-transform $H_0(z)$, analysis highpass filter $H_1(z)$ and with synthesis filters $G_0(z)$ and $G_1(z)$ is shown by the diagram below (Fig. 1) .

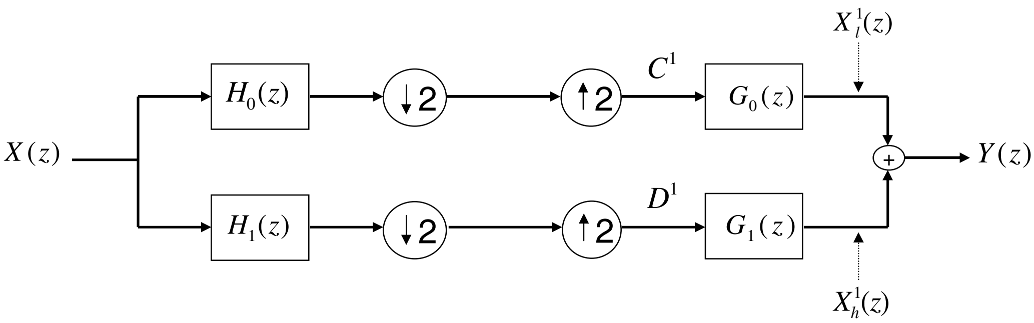

Figure 1 - DWT filter bank

For an input signal $X(z)$, the analysis part of the filter bank followed by upsampling produces the low-pass (eq. (1)) and the high-pass (eq. (2)) coefficients

$$C^1(z^2) = \frac{1}{2}\{X(z)H_0(z) + X(-z)H_0(-z)\} \tag{1}$$

$$D^1(z^2) = \frac{1}{2}\{X(z)H_1(z) + X(-z)H_1(-z)\} \tag{2}$$

respectively, and decomposes the input signal into a low frequency part $X_l^1(z)$ and a high frequency part $X_h^1(z)$, the output signal is the sum of these two components (eq. (3))

$$Y(z) = X_l^1(z) + X_h^1(z) \tag{3}$$

where

$$X_l^1(z) = C^1(z^2)G_0(z) = \frac{1}{2}\{X(z)H_0(z)G_0(z) + X(-z)H_0(-z)G_0(z)\} \tag{4}$$

$$X_h^1(z) = D^1(z^2)G_1(z) = \frac{1}{2}\{X(z)H_1(z)G_1(z) + X(-z)H_1(-z)G_1(z)\} \tag{5}$$

This decomposition is not shift invariant due to the terms in $X(-z)$ of (eq. (4)) and (eq. (5)), respectively, which are introduced by the downsampling operators. If the input signal is shifted, for example $z^{-1}X(z)$, the application of the filter bank results in the decomposition (eq. (6))

$$z^{-1}X(z) = \tilde{X}_l^1(z) + \tilde{X}_h^1(z) \tag{6}$$

For an input signal $z^{-1}X(z)$ we have

$$C^1(z^2) = \frac{1}{2}\{z^{-1}X(z)H_0(z) + (-z^{-1})X(-z)H_0(-z)\} \tag{7}$$

and

$$\tilde{X}_l^1(z) = \frac{1}{2}z^{-1}\{X(z)H_0(z)G_0(z) - X(-z)H_0(-z)G_0(z)\} \tag{8}$$

and similarly for the high-pass part, which of course is not the same as $z^{-1}X_l^1(z)$ if we substitute for z^{-1} in eq. (4). From this calculation it can be seen that the shift dependence is caused by the terms containing $X(-z)$, the *aliasing terms*.

One possibility to obtain a shift invariant decomposition can be achieved by the addition of a filter bank to Fig. 1 with shifted analysis filters $z^{-1}H_0(z)$, $z^{-1}H_1(z)$ and synthesis filters $zG_0(z)$, $zG_1(z)$ and subsequently taking the average of the lowpass and the highpass branches of both filter banks (Fig. 2).

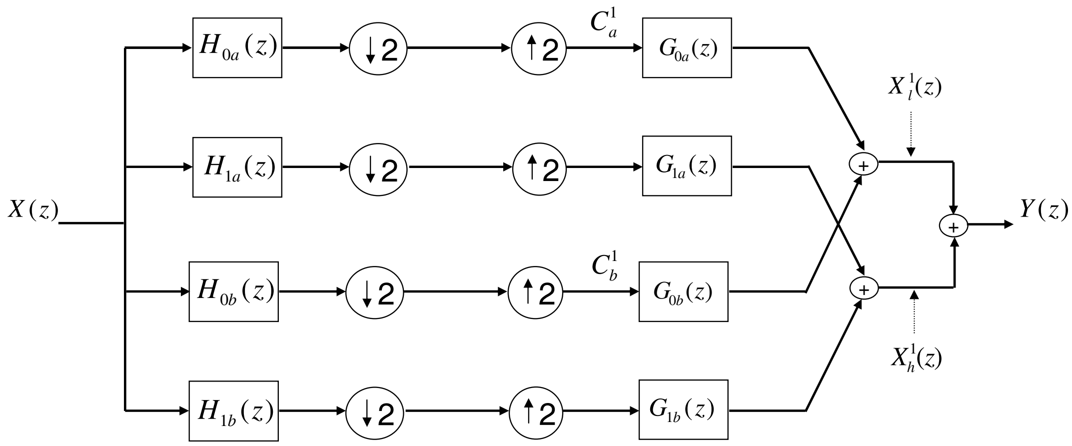

Figure 2 - One level complex dual tree

If we denote the first filter bank by index a and the second one by index b then this procedure implies the following decomposition (eq. (9))

$$X(z) = X_l^1(z) + X_h^1(z) \tag{9}$$

where for the lowpass channels of tree a and tree b we have

$$\begin{aligned} X_l^1(z) &= \frac{1}{2}\{C_a^1(z^2)G_{0a}(z) + C_b^1(z^2)G_{0b}(z)\} \\ &= \frac{1}{4}\{[X(z)H_0(z) + X(-z)H_0(-z)]G_0(z)+ \\ &\quad + [X(z)z^{-1}H_0(z) + X(-z)(-z^{-1})H_0(-z)]zG_0(z)\} \\ &= \frac{1}{4}\{X(z)[H_0(z)G_0(z) + H_0(z)G_0(z)]+ \\ &\quad + X(-z)[H_0(-z)G_0(z) - H_0(-z)G_0(z)]\} \\ &= \frac{1}{2}X(z)H_0(z)G_0(z) \end{aligned} \tag{10}$$

and similarly for the high-pass part. The aliasing term containing $X(-z)$ in X_l^1 has vanished and the decomposition becomes indeed shift invariant.

Using the same principle for the design of shift invariant filter decomposition, Kingsbury suggested in [2] to apply a 'dual-tree' of two parallel filter banks and combine their bandpass outputs. The structure of a resulting analysis filter bank is sketched in Fig. 3, where index a stands for the original filter bank and the index b is for the additional one. The dual-tree complex DWT of a signal $x(n)$ is implemented using two critically-sampled DWTs in parallel on the same data.

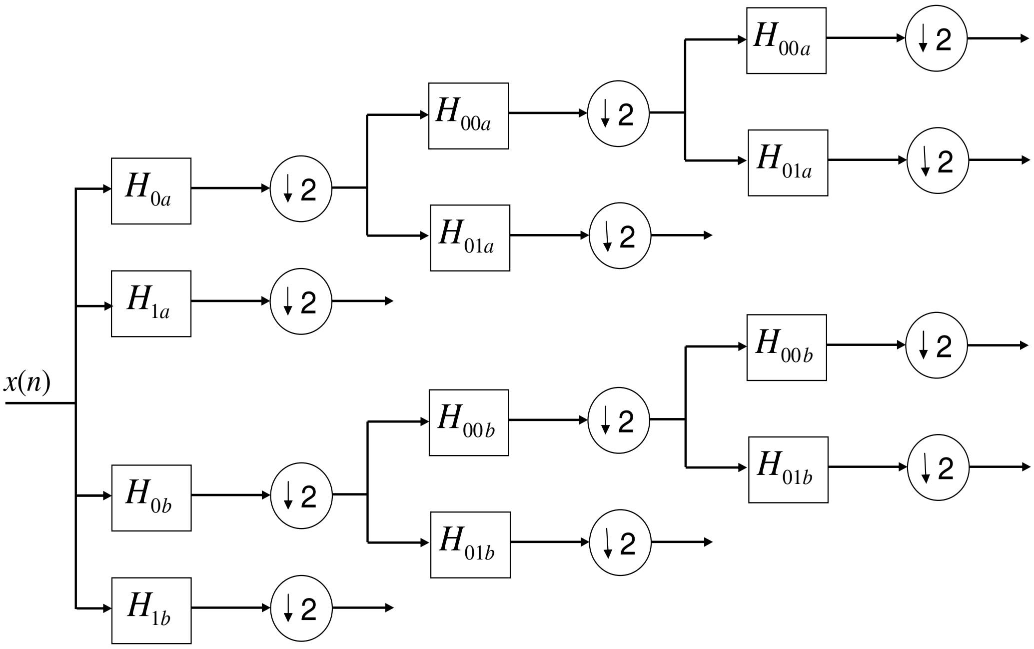

Figure 3 - Complex dual tree

In one dimension, the so-called dual-tree complex wavelet transform provides a representation of a signal $x(n)$ in terms of complex wavelets, composed of real and imaginary parts which are in turn wavelets themselves. In fact, these real and imaginary parts essentially form a quadrature pair [8], the complex wavelet associated with the dual tree CDWT is shown below (Fig. 4).

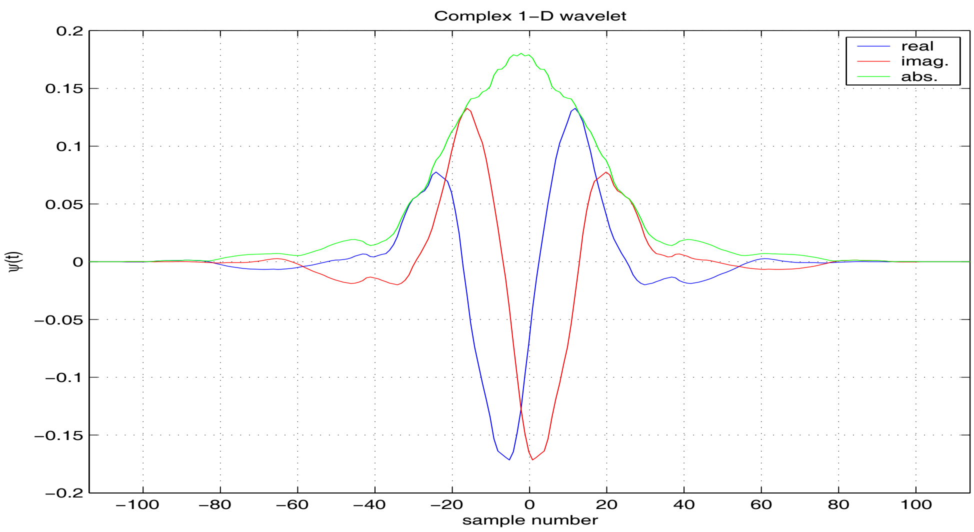

Figure 4 - Impulse response of a complex wavelet

The dual-tree CDWT uses length-10 filters [4], the table of coefficients of the analysing filters in the first stage (Tab. 1) and the remaining levels (Tab. 2) are shown. The reconstruction filters are obtained by simply reversing the alternate coefficients of the analysis filters.

Table 1 - First level DWT coefficients

Tree a		*Tree b*	
H_{0a}	H_{1a}	H_{0b}	H_{1b}
0	0	0.01122679	0
-0.08838834	-0.01122679	0.01122679	0
0.08838834	0.01122679	-0.08838834	-0.08838834
0.69587998	0.08838834	0.08838834	-0.08838834
0.69587998	0.08838834	0.69587998	0.69587998
0.08838834	-0.69587998	0.69587998	-0.69587998
-0.08838834	0.69587998	0.08838834	0.08838834
0.01122679	-0.08838834	-0.08838834	0.08838834
0.01122679	-0.08838834	0	0.01122679
0	0	0	-0.01122679

Table 2 - Remaining levels DWT coefficients

Tree a		*Tree b*	
H_{00a}	H_{01a}	H_{00b}	H_{01b}
0.03516384	0	0	-0.03516384
0	0	0	0
-0.08832942	-0.11430184	-0.11430184	0.08832942
0.23389032	0	0	0.23389032
0.76027237	0.58751830	0.58751830	-0.76027237
0.58751830	-0.76027237	0.76027237	0.58751830
0	0.23389032	0.23389032	0
-0.11430184	0.08832942	-0.08832942	-0.11430184
0	0	0	0
0	-0.03516384	0.03516384	0

To extend the transform to higher-dimensional signals, a filter bank is usually applied separably in all dimensions. To compute the 2D CWT of images these two trees are applied to the rows and then the columns of the image as in the basic DWT. This operation results in six complex high-pass subbands at each level and two complex low-pass subbands on which subsequent stages iterate in contrast to three real high-pass and one real low-pass subband for the real 2D transform. This shows that the complex transform has a coefficient redundancy of 4:1 or $2^m : 1$ in m dimensions. In case of real 2D filter banks the three highpass filters have orientations of $0^0, 45^0$ and 90^0, for the complex filters the six subband filters are oriented at $\pm 15^0, \pm 45^0, \pm 75^0$ (Fig. 5).

Figure 5 - Complex filter response showing the orientations of the complex wavelets

The CDWT decomposes an image into a pyramid of complex subimages, with each level containing six oriented subimages resulting from evenly spaced directional filtering and subsampling, such directional filters are not obtainable by a separable DWT using a real filter pair but complex coefficients makes this selectivity possible.

3 RESULTS

The shift invariance and directionality of the CWT may be applied in many areas of image processing like denoising, feature extraction, object segmentation and image classification. Here we shall consider the denoising example. For denoising a soft thresholding method is used. The choice of threshold limits λ for each decomposition level and modification of the coefficients is defined by eq. (11).

$$\bar{c}_s(k) = \begin{cases} \text{sign } c(k)(|c(k)| - \lambda) & \text{if } |c(k)| > \lambda \\ 0 & \text{if } |c(k)| \leq \lambda \end{cases} \tag{11}$$

To compare the efficiency of the DWT with the basic DWT the quantitative mean square error (MSE) is used. In all cases the optimal thresholds points λ were selected to give the minimum square error from the original image (Fig. 7).

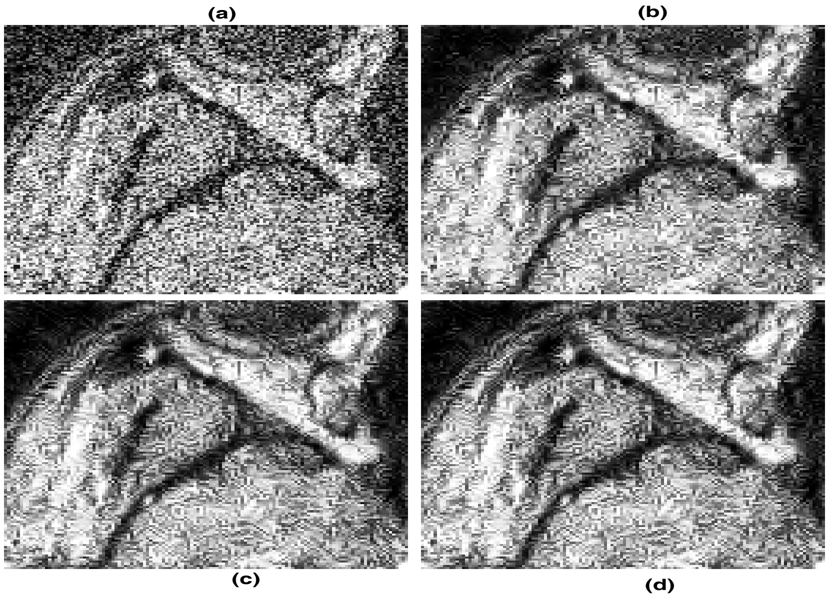

Figure 6 - MRI image scan: (a) with random noise added, (b) denoised with DWT, (c) denoised with real CWT, (d) denoised with dual tree CWT

From Fig. 6(b) it may be seen that DWT introduces prominent worse artifacts, while the DT CWT provides a qualitatively restoration with a better optimal minimum MSE error.

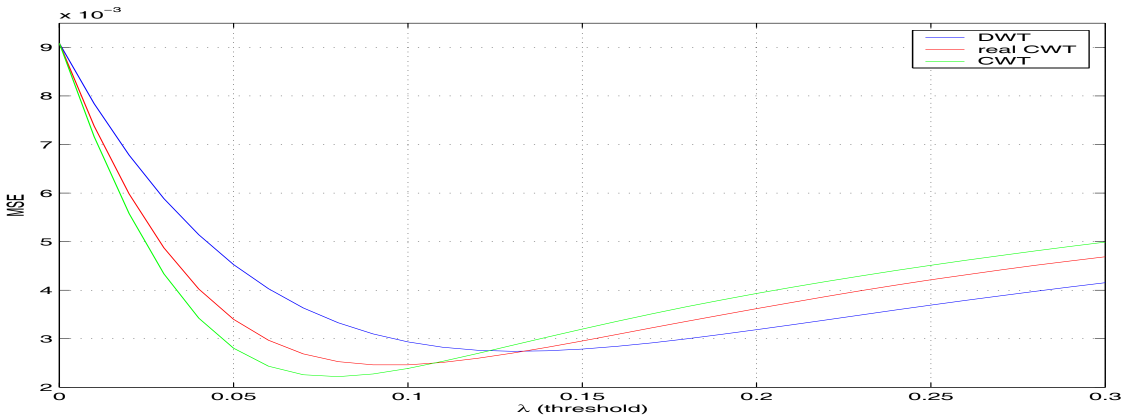

Figure 7 - Optimal threshold points for the three different methods

4 CONCLUSION

The DT CWT is shift invariant and forms directionally selective diagonal filters. These properties are important for many applications in image processing including denoising, deblurring, segmentation and classification. In this paper we have illustrated the example of the application of complex wavelets for the denoising of MR images, showing a great effectiveness in removing the noise compared to the classical DWT as Tab. 3 shows.

Table 3 - MEAN SQUARE ERROR (MSE) AND SIGNAL-TO-NOISE RATIO (SNR) VALUES

Type of method	MSE	SNR [dB]
noisy image	0.0418	20.8347
DWT	0.0262	25.4986
real CWT	0.0255	25.7601
CWT	0.0240	26.3751

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ACKNOWLEDGMENTS

The work has been supported by the research grant of the Faculty of Chemical Engineering of the Institute of Chemical Technology, Prague No. MSM 223400007.

The MATLAB programs used for the implementation of the dual-tree CWT were obtained at *http://taco.poly.edu/WaveletSoftware*