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ALGORITHMIC APPROACH TO DIGITAL SIGNAL PROCESSING

Principles, Methods and Applications

Digital Signal and Image Processing Research Group

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Introduction

Digital signal processing (DSP) represents a general interdisciplinary area [36, 30, 3, 41, 35] based upon numerical or symbolical analysis of one-dimensional or multi-dimensional data sets that may stand for any physical, engineering, biomedical, technological, biological, acoustic, seismic or economical variable measured or observed with a given sampling period. Selected applications and goals of their processing are presented in Fig. 1.1. Even though applications cover completely different areas the mathematical background of their analysis is very close allowing processing of vectors, matrices or multi-dimensional arrays of observed data in a general way. Digital signal processing methods thus form an integrating platform for many diverse research branches.

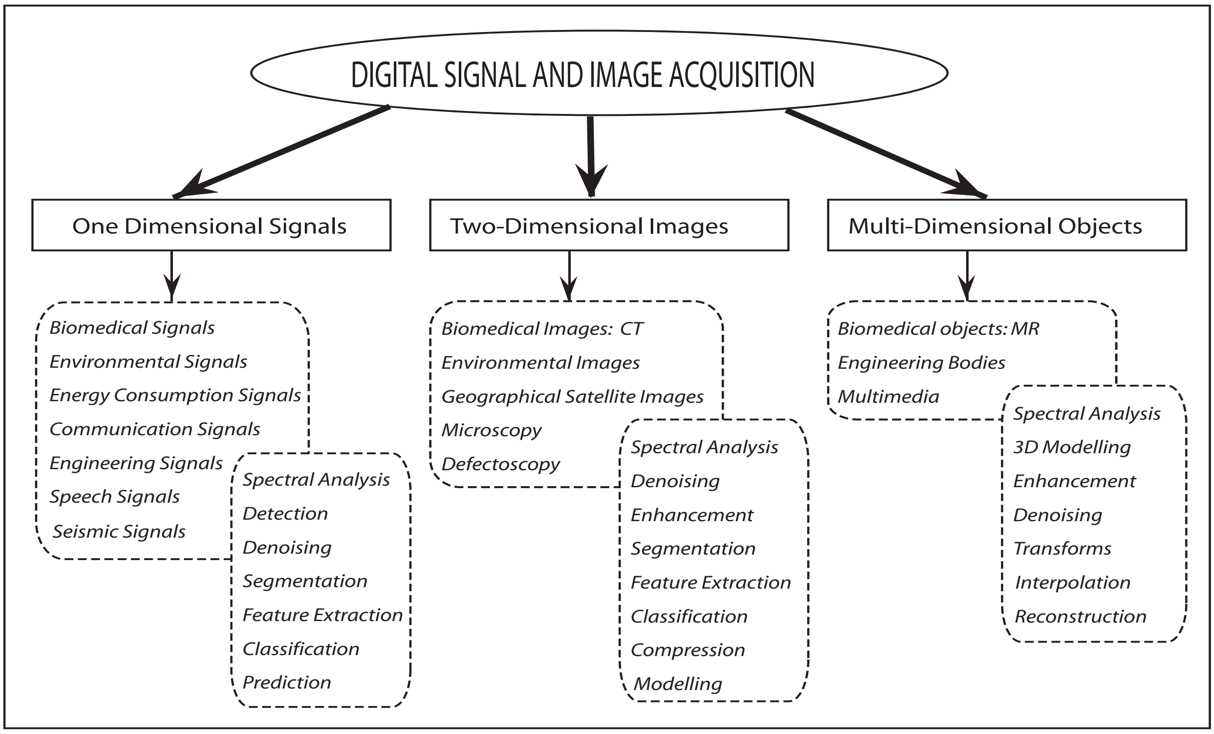

FIGURE 1.1. Fundamental applications of one-dimensional and multi-dimensional signal processing and selected goals of their analysis

Fundamental mathematical methods of signal, image and multi-dimensional objects processing in the space and frequency domains are summarized in Fig. 1.2 and they include the following main topics

- Space domain deterministic processing
- Probabilistic signal processing
- Space domain adaptive processing
- Space-Frequency analysis
- Space-Scale analysis

Selected mathematical methods presented in this figure cover basic numerical methods [4], statistical methods, adaptive methods including neural networks, discrete Fourier transform and discrete wavelet transform. Goals of signal analysis cover the estimation of its characteristic parameters either in the *original* or in the *transform* domain. In some cases of signal processing *deterministic methods* may be applied but in many applications *statistical* and *adaptive methods* [12, 33] must be used to compensate for the incomplete knowledge of the real system time variations.

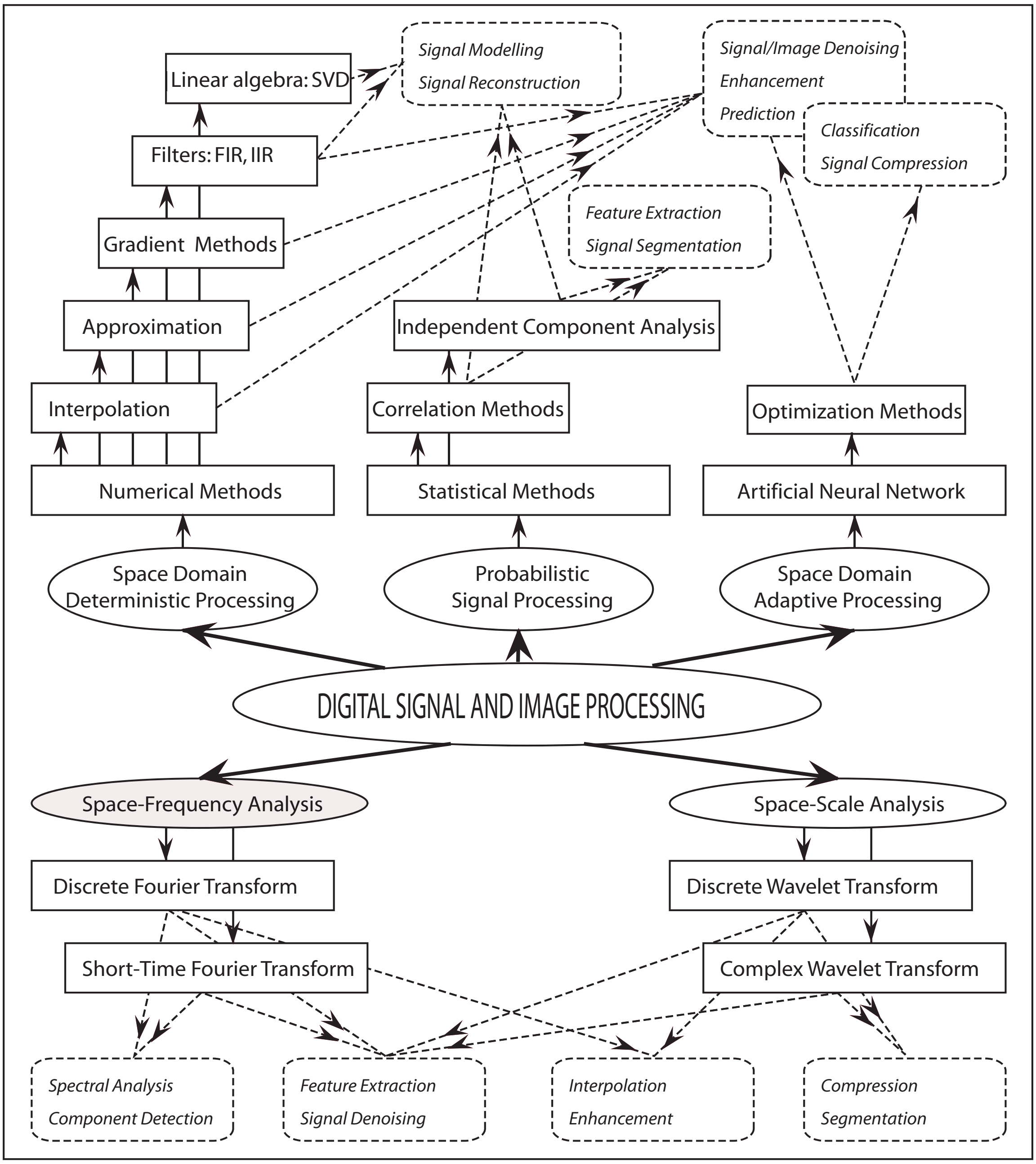

FIGURE 1.2. Fundamental mathematical methods of digital signal and image processing both in the space and frequency domains and associated goals of their applications

1.1 Historical Notes

The mathematical fundamentals of digital signal and image processing methods are based upon *numerical analysis* [1] that predates the invention of modern computers by many centuries using works of famous mathematicians including that of Isaac Newton (1643-1727), Joseph Louis Lagrange (1736-1813) and Leonhard Euler (1707-1783). The *matrix theory* introduced in the middle of the 19th century incorporating ideas of Gottfried Wilhelm Leibnitz (1646-1716) and Carl Friedrich Gauss (1777-1855) forms now one of its basic mathematical tools as well.

Isaac Newton
(January 1,1643-March 31,1727)

Jean Baptiste Joseph Fourier
(March 21,1768-May 16,1830)

Carl Friedrich Gauss
(April 30,1777-February 23,1855)

The theory of digital signal processing is in many cases closely connected with the *Fourier representation* of functions suggested in 1822 [9] by Jean Baptiste Joseph Fourier (1786-1830) and on the *method of the least squares* presented independently by Carl Friedrich Gauss [10] and Adrien-Marie Legendre (1752-1833) [21] in the beginning of the 19th century.

Basic mathematical methods were later extended to many fields including the estimation theory and stochastic processes introduced by Norbert Wiener (1894-1964) in 1949 [44] and Rudolf E. Kalman (1930-) [19] with applications in various areas covering adaptive filtering problems and spectrum analysis. Many algorithms use properties of the discrete Fourier transform and their implementation is enabled by its fast version published by James Cooley (1926-) and John Tukey (1915-2000) [7] in 1965. The latest research of wavelet transform using the first known wavelet proposed by Alfred Haar (1885-1933) in 1909 is based upon research of Ingrid Daubechies (1954-) published in 1992 [8].

Adrien-Marie Legendre
(September 18,1752-January 10,1833)

1.2 Applications

Basic methods and applications of digital signal and image processing methods summarized in Fig. 1.1 cover [40, 12, 3, 41, 14]

- *Spectrum estimation* giving the distribution of power over frequency and enabling in this way to distinguish characteristic signal components important for analysis and further processing [13] in biomedicine, chemistry, seismology, communications and control systems,
- *Digital filtering* originally used to eliminate undesirable frequency components or to reduce noise in communications, multimedia systems, control structures or biomedical data [29, 16, 17] and to improve possibilities of acoustic and image processing,
- *Correlation techniques* enabling comparison of signals representing in many cases sampled physical quantity.

Classical applications mentioned above has been substantially enlarged by the use of adaptive filters giving ability to operate in an unknown environment on discrete signals representing any physical or technological variable. General objectives of such a processing may result in

- *System identification* substantial for mathematical modelling in science or engineering and including *parameter estimation* and *inverse modelling* as well,
- *Signal detection* and *prediction* enabling to find out the useful information in a given sequence and to forecast its behaviour,
- *Interference cancelling* used to reject undesirable signal components for further signal analysis and processing.

In case of general systems time-varying models are obtained and methods used for adaptive processing are closely related to that of computational intelligence and neural networks.

1.3 Algorithmic Tools

Computer experiments suggested in this text may be transformed into any computer language but all presentations described bellow were realized in the MATLAB software package [11, 26, 27, 6] allowing a very simple realization of all methods in form very close to the original matrix notation. There are many books devoted to signal and image processing using this computational environment [38, 14, 37, 28].

2

Discrete Signals and Systems

Let us start the study of digital signal processing methods by the summary of basic signals and systems properties and mathematical tools enabling their description and analysis. Such a background is substantial for the development of classical and adaptive methods described further.

2.1 Fundamental Concepts

Most of observed *signals* are *continuous functions* $x_a(\mathbf{t})$ of one or more variables. Before their digital processing it is necessary to realize their *sampling* with a given sampling period T_s (or sampling frequency $f_s = 1/T_s$). In case of one independent variable (usually standing for time) resulting *discrete-time signal* is represented (Fig. 2.1) by a sequence of numbers

$$\mathbf{x} = \{x(n)\} = \{x_a(nT_s)\} \tag{2.1}$$

for $n \in (-\infty, +\infty)$. As real analog/digital converters are able to approximate discrete-time values by a limited number of digits only such a sequence is *digital* in fact [30, 23].

Time domain signal description enables definition of *deterministic signals* including *periodic* and *nonperiodic signals* by their mathematical definition. The most important deterministic signals represent

- unit sample sequence: $d(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$
- unit step sequence: $u(n) = \begin{cases} 1 & \text{for } n \geq n_0 \\ 0 & \text{for } n < n_0 \end{cases}$
- real exponential sequence: $x(n) = a^n$
- sinusoidal sequence: $x(n) = A\, sin(2\pi f n)$

The sketch of these signals is given in Fig. 2.2.

Further signals may be described by their own mathematical definition and they may be also combined using the basic operations summarized in Tab. 2.1 (including also MATLAB notation which in real programs does have no formal difference between scalars, vectors or matrixes considering a scalar as a special matrix with one element only).

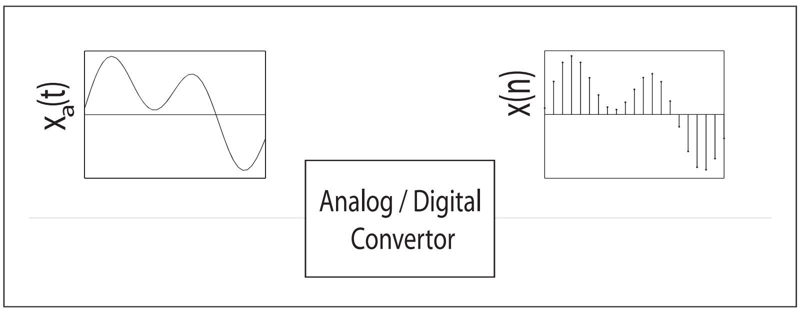

FIGURE 2.1. Sampling process of an analog signal

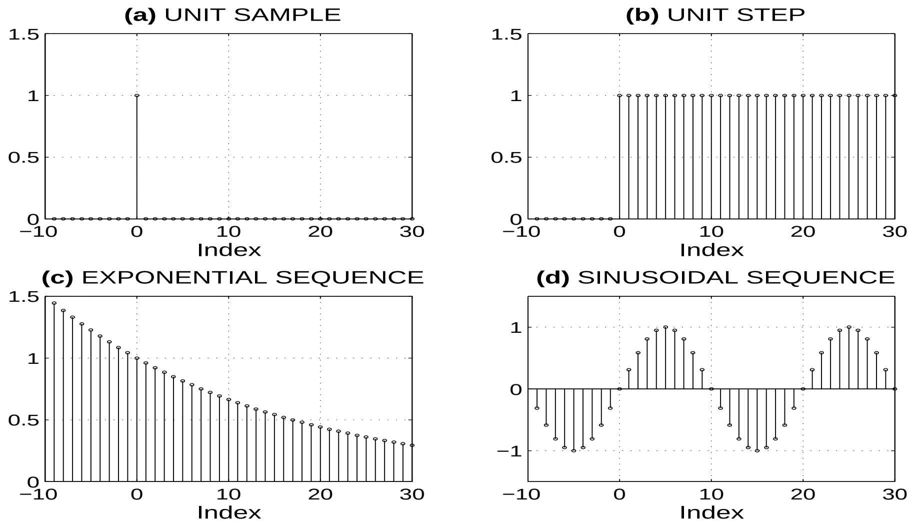

FIGURE 2.2. Basic deterministic signals

In many practical cases observed signals are *random* including unpredictable noise as well. Description of such signals is based upon a random signal theory presented in many books including [31, 2, 12]. The analysis of these signals may be in many cases restricted to *stationary random signals* with their basic probabilistic characteristics (average and autocovariance function) independent of the starting index of observation. An example of a random signal with its histogram approximating its probabilistic distribution is given in Fig. 2.3.

For various signal analysis techniques it is useful to refer to the *energy* of a sequence defined ([23, p.24] or [30, p.10]) as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.2}$$

Frequency domain signal description is another method of the given sequence approximation which is substantial in many digital signal processing methods. Fourier series applied for continuous signals studied for example in [24, p.10] or [23, p.258] are originally restricted to the approximation of a periodic function $f(t)$ with period T by the weighted sum of complex exponentials or trigonometric functions in the form

$$f_{aprox}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\frac{2\pi}{T}t} = a_0 + \sum_{k=1}^{\infty}\left(a_k cos(k\frac{2\pi}{T}t) + b_k sin(k\frac{2\pi}{T}t)\right) \tag{2.3}$$

Operation	*Definition*	*MATLAB notation*
multiplication	$\{x(n)\}\cdot\{y(n)\} \equiv \{x(n)\cdot y(n)\}$	**x** .⋆ **y**
linear combination	$a\cdot\{x(n)\} + b\cdot\{y(n)\} \equiv \{a\cdot x(n) + b\cdot y(n)\}$	a⋆**x**+b⋆**y**
convolution	$\{x(n)\} \star \{y(n)\} \equiv \{\sum_{k=-\infty}^{\infty} x(k)y(n-k)\}$	conv(**x**,**y**)
translation	$\{x(n-n_0)\}$	

TABLE 2.1. Basic sequence operations

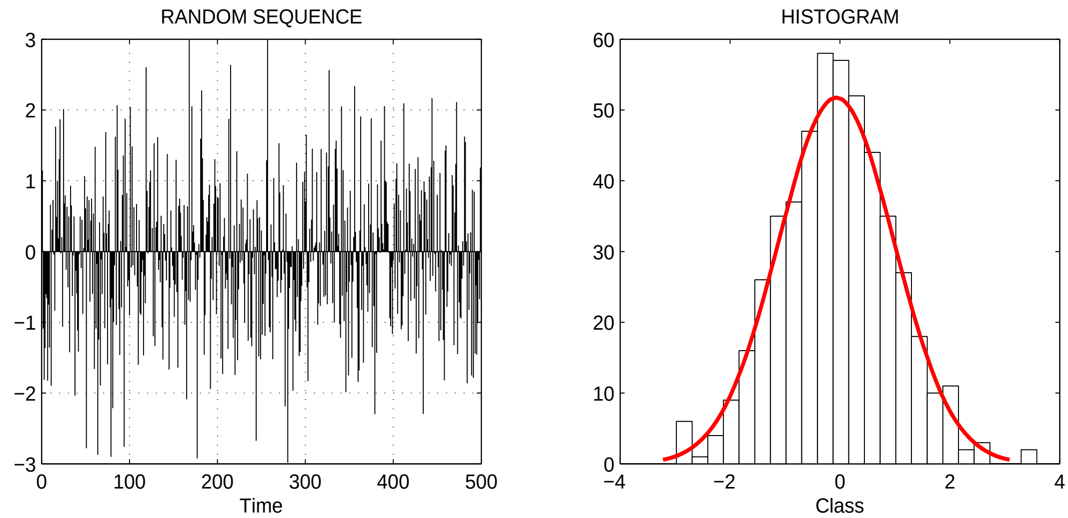

FIGURE 2.3. Stationary random signal with normal probabilistic distribution and its histogram

Using the mean square error method it is possible to derive that

$$F_k = \frac{1}{T} \int_0^T e^{-jk\frac{2\pi}{T}t} \, dt \tag{2.4}$$

for $k = 0, \pm 1, \pm 2, \cdots$ and after the application of Euler relations for complex exponentials it is possible to express

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T f(t) \, dt \\ a_k &= \frac{1}{T} \int_0^T f(t) \, cos(k\frac{2\pi}{T}t) \, dt \\ b_k &= \frac{1}{T} \int_0^T f(t) \, sin(k\frac{2\pi}{T}t) \, dt \end{aligned} \tag{2.5}$$

for $k = 1, 2, \cdots$ .Example of such an approximation of a rectangular function with its period $T = 2\pi$ by a limited number of terms in the form

$$f_{aprox}(t) = \frac{4}{\pi}(sin(t) + \frac{1}{3}sin(3t) + \frac{1}{5}sin(5t) + \frac{1}{7}sin(7t)) \tag{2.6}$$

is presented in Fig. 2.4 together with weights denoting the significance of separate frequency components. Generalization of this method to non-periodic signals is studied further in connection with the *discrete Fourier transform* analysed for instance in [40, p.59] as well. It enables signal description in the form of a finite number of its frequency components giving possibility of the sampling rate estimation as well.

Theorem 2.1 *Let f_m is the highest frequency component of a signal. Then the sampling frequency f_s must be greater or equal then $2f_m$ to enable its perfect reconstruction.*

Proof of this theorem is closely connected with the theory of the discrete Fourier transform presented further and studied in many books including [30, p.28], [40, p.45] or [23, p.57].

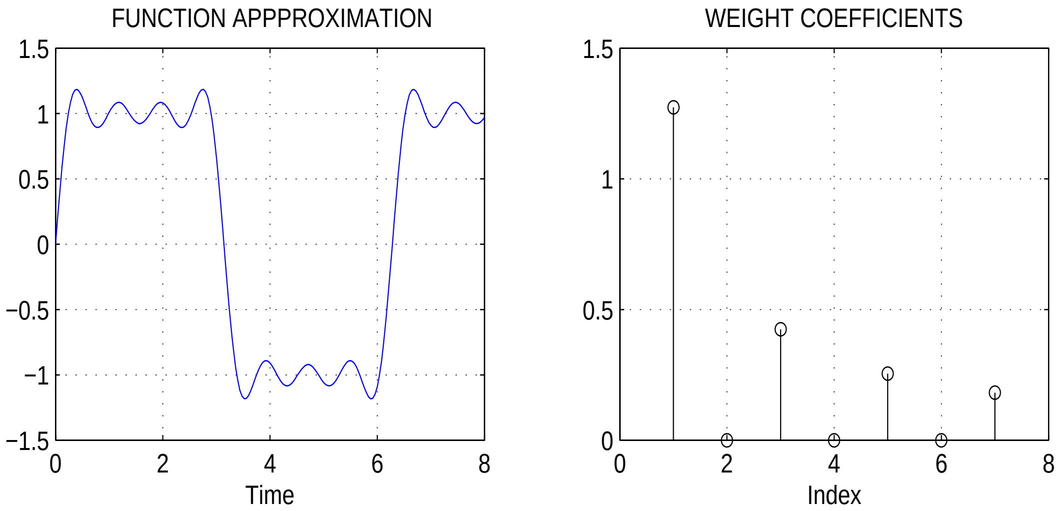

FIGURE 2.4. Rectangular function approximation with the separate frequency components

2.2 Discrete System Description

A *discrete system* is mathematically defined as a transform of the input sequence $\{x(n)\}$ into the output sequence $\{y(n)\}$ by means of an operator $\mathbf{T}$ (Fig. 2.5). In case of the unit sample input sequence $\{d(n)\}$ the system output is called the *impulse response* $\{h(n)\}$ having substantial role in signal analysis presented further. Process of such a transformation is often called *digital filtering* which in the broader sense includes both extraction of information from a given signal and system identification or control as well.

Definition 2.1 *Linear shift invariant system is a discrete system having the following properties*

$$\mathbf{T}[a\, x_1(n) + b\, x_2(n)] = a\, \mathbf{T}[x_1(n)] + b\, \mathbf{T}[x_2(n)] \tag{2.7}$$

$$\mathbf{T}[x(n)] = y(n) \Rightarrow \mathbf{T}[x(n-k)] = y(n-k) \tag{2.8}$$

Theorem 2.2 *Let $\{h(k) : h(k) = \mathbf{T}[d(k)]\}$ stands for the impulse response of a discrete linear shift invariant system. Then the response of this system to the signal $\{x(n)\}$ is determined by the convolution sum*

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \tag{2.9}$$

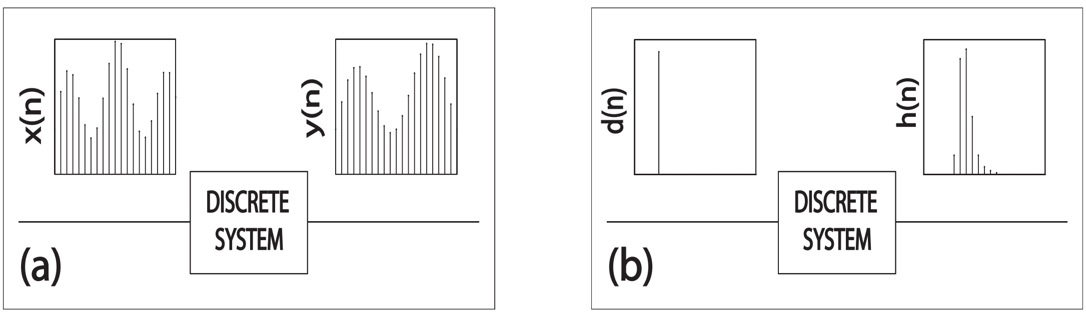

FIGURE 2.5. General discrete system and its application for impulse processing

Proof: It is obvious that it is possible to define signal $\{x(n)\}$ by means of the impulse function $\{d(n)\}$ in the form

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\ d(n-k)$$

Using system operator T it is possible to calculate the system output

$$y(n) = \mathbf{T}[\sum_{k=-\infty}^{\infty} x(k)\ d(n-k)]$$

After application of properties (2.7) and (2.8) of a shift invariant system it is possible to write

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k)\ \mathbf{T}[d(n-k)] = \\ &= \sum_{k=-\infty}^{\infty} x(k)\ h(n-k) \end{aligned}$$

By further substitution to change indices we shall receive expression (2.9). △

Results presented above implies that any linear shift invariant system is completely defined by its unit sample response $\{h(n)\}$. This result can be further used to determine *system stability* [23, p.34].

Definition 2.2 *A discrete system is said to be stable if every bounded input sequence* $\{x(n)\}$ *implies bounded output sequence* $\{y(n)\}$.

Theorem 2.3 *A linear shift invariant system is stable if and only if the sum*

$$S = \sum_{k=-\infty}^{\infty} |h(k)| \tag{2.10}$$

has a finite value.

Proof: Assume that the input sequence is bounded by a finite M such that $|x(n)| < M$ for all n. Then it is possible to use Eq. (2.9) and the triangular inequality to write

$$|y(n)| = |\sum_{k=-\infty}^{\infty} h(k)\ x(n-k)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\ |x(n-k)| < M \sum_{k=-\infty}^{\infty} |h(k)| = M\ S$$

Therefore if S is finite the output sequence is bounded as well. △

Further considerations are in most cases restricted to *causal systems* [23, p.38] having their output for each n dependent on input values for $k \leq n$ only. Impuls response $\{h(n)\}$ of such systems is nonzero for $n \geq 0$ only and Eq. (2.9) has therefore the following form

$$y(n) = \sum_{k=0}^{\infty} h(k)\ x(n-k) \tag{2.11}$$

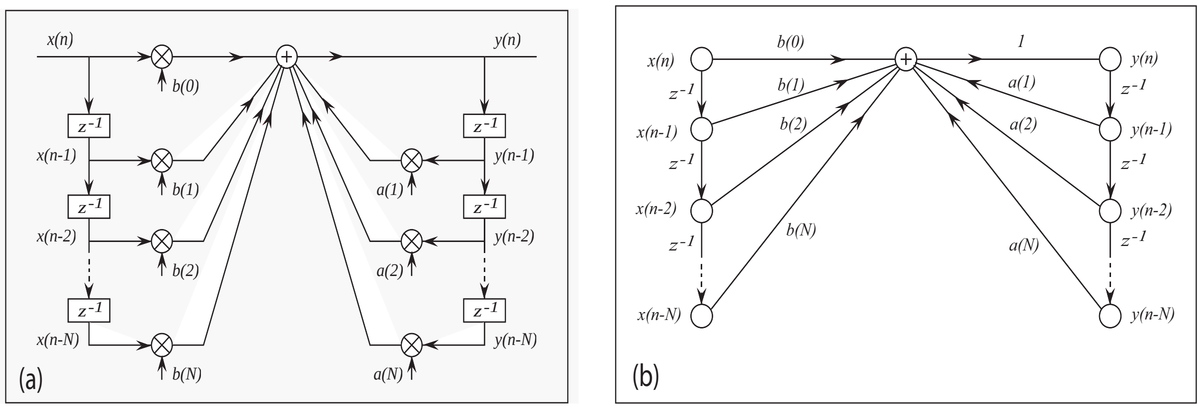

FIGURE 2.6. Block diagram and signal flow graf representation of the IIR filter (with symbol z^{-1} standing for the unit sample delay)

TIME DOMAIN DISCRETE SYSTEM DESCRIPTION may be in many cases restricted to the *linear constant coefficient difference equation* ([40, p.180], [30, p.16]) defining relationship between the input and output sequence in the form

$$y(n) + \sum_{k=1}^{N} a(k)\ y(n-k) = \sum_{k=0}^{N} b(k)\ x(n-k) \tag{2.12}$$

This general equation denoted as *autoregressive-moving average* (ARMA) *model* can take the following specific simplifications

(i) *moving average* (MA) *model* in the form

$$y(n) = \sum_{k=0}^{N} b(k)\ x(n-k) \tag{2.13}$$

(ii) *autoregressive* (AR) *model* in the form

$$y(n) + \sum_{k=1}^{N} a(k)\ y(n-k) = x(n) \tag{2.14}$$

Comparing Eqs. (2.13) and (2.11) it is possible to see that coefficients $\{b_0, \cdots, b_N\}$ stand for the finite duration impulse response $\{h_0, \cdots, h_N\}$ and corresponding digital system is therefore also called *finite impulse response* (FIR) *filter*. Owing to Theorem 2.2 it is always stable which explains one of reasons of its popularity.

General autoregressive and autoregressive-moving average model represent *infinite impulse response* (IIR) *filter* as explained in the following example. Graphical description of such a general system in the *block diagram* form and *signal flow graf representation* [30, p.136] is presented in Fig. 2.6.

Example 2.1 Calculate the unit sample response of a digital system described by the difference equation (2.12).

Solution: Assume the input sequence $\{x(n)\} = \{d(n)\}$. Denoting the impulse response $\{y(n)\} = \{h(n)\}$ it is possible to use Eq. (2.12) to evaluate sequence $\{h(n)\}$ in the form

$$\begin{aligned}
h(n) &= 0, \qquad n < 0 \\
h(0) &= b_0 \\
h(1) &= b_1 - a_1\, h(0) \\
h(2) &= b_2 - a_1\, h(1) - a_2\, h(0) \\
&\cdots \\
h(N) &= b_N - \sum_{k=1}^{N} a_k\, h(N-k) \\
h(n) &= -\sum_{k=1}^{N} a_k\, h(n-k), \qquad n > N
\end{aligned}$$

This generally infinite sequence stands for the infinite impulse response of studied digital system.

General shift invariant model of the linear shift invariant system described by the difference Eq. (2.12) can be expressed in the following vector form

$$y(n) + \mathbf{a} \begin{bmatrix} y(n-1) \\ \vdots \\ y(n-N) \end{bmatrix} = \mathbf{b} \begin{bmatrix} x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix} \tag{2.15}$$

where $\mathbf{a}=[a_1, \cdots, a_N]$
$\mathbf{b}=[b_1, \cdots, b_N]$

This system representation involves calculations with past values of the signal output variables.

State space representation of a digital filter described for instance in [18, p.84] or [23, 231] enables evaluation of the output value $y(n)$ as a linear combination of the input value $x(n)$ and *state variables* $\mathbf{v}\ (n) = [v_1(n), \cdots, v_N(n)]'$ in the form

$$y(n) = \mathbf{c}\,\mathbf{v}(n) + d\,x(n) \tag{2.16}$$

where $\mathbf{c} = [c_1, \cdots, c_N], d$ are the state space model coefficients. The state space vector of the system represents the minimal information required to determine the output and it must be updated for each n for a linear discrete system by state equation

$$\mathbf{v}(n+1) = \mathbf{A}\,\mathbf{v}(n) + \mathbf{b}\,x(n) \tag{2.17}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ & \cdots & \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \tag{2.18}$$

stand for so called state transition matrix and excitation vector respectively.

State space method can be simply applied for multiple inputs and outputs as well (using vectors and matrices instead of scalars and vectors) and can be also used for time varying systems. Graphical description of a general state space representation of a discrete system in signal flow graf notation is presented in Fig. 2.7.

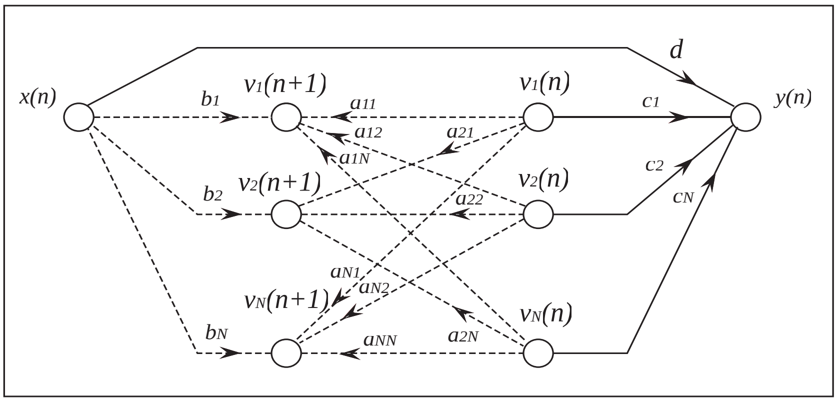

FIGURE 2.7. Signal flow graf representation of the state space system description

Example 2.2 *Derive space equations for the FIR system described by the difference equation in the form*

$$y(n) = h_0\, x(n) + h_1\, x(n-1) + \cdots + h_N\, x(n-N)$$

Solution: Let us define the first state variable

$$v_N(n) = h_1\, x(n-1) + h_2\, x(n-2) + \cdots + h_N\, x(n-N)$$

allowing to express the output equation in the form

$$y(n) = [0\,0 \cdots 1]\, \mathbf{v}(n) + h_0\, x(n) \tag{2.19}$$

To derive further state equations let us express

$$\begin{aligned} v_N(n+1) &= h_1\, x(n) + h_2\, x(n-1) + \cdots + h_N\, x(n+1-N) = \\ &= h_1\, x(n) + v_{N-1}(n) \end{aligned}$$

with the next state variable in the form

$$v_{N-1}(n) = h_2\, x(n-1) + h_3\, x(n-2) + \cdots + h_N\, x(n+1-N)$$

In the same way it is possible to derive

$$\begin{aligned} v_{N-1}(n+1) &= h_2\, x(n) + v_{N-2}(n) \\ v_{N-2}(n+1) &= h_3\, x(n) + v_{N-3}(n) \\ &\cdots \\ v_2(n+1) &= h_{N-1}\, x(n) + v_1(n) \\ v_1(n+1) &= h_N\, x(n) \end{aligned}$$

and to write the state equation in the form

$$\mathbf{v}(n+1) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} h_N \\ h_{N-1} \\ \vdots \\ h_2 \\ h_1 \end{bmatrix} x(n) \tag{2.20}$$

FREQUENCY DOMAIN REPRESENTATION of a linear shift invariant system ([30, p.19] or [23, p.84]) is very useful in the linear system theory as it provides information about signal processing with respect to its frequency components. In particular the steady state response of such a system to the sinusoidal function is a sinusoid of the same frequency but different amplitude and phase determined by the system.

Since a sinusoid can be defined by the sum of two complex exponentials we can apply a discrete input sequence in the form

$$x(n) = e^{j\omega n}$$

Using Theorem 2.2 it is possible to determine system output in the form

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\ e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\ e^{-j\omega k}$$

Defining the *frequency response*

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\ e^{-j\omega k}$$

we can evaluate the system output

$$y(n) = H(e^{j\omega})\ e^{j\omega n} \tag{2.21}$$

Using magnitude and phase of the frequency response it is further given by expression

$$y(n) = |H(e^{j\omega})|\ e^{arg(H(e^{j\omega}))}\ e^{j\omega n}$$

Result presented by Eq. (2.21) is valid under assumption that the input sequence has been applied for $k \to -\infty$. In real applications the discrete time system provides *transient period* before the *steady state* response.

Example 2.3 *Evaluate amplitude frequency response of a moving average system described by equation*

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

Solution: Applying $x(n) = e^{j\omega n}$ we shall receive

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\omega(n-k)} = e^{j\omega n} \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\omega k}$$

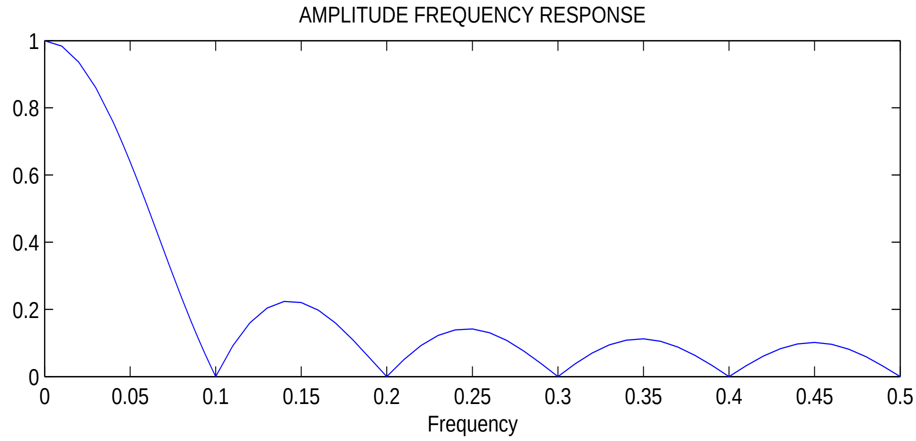

FIGURE 2.8. Amplitude frequency response of the moving average discrete system

Evaluating the sum of the geometrical sequence we shall obtain

$$y(n) = e^{j\omega n}\frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

As

$$H(e^{j\omega}) = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{1}{N}\frac{e^{-j\omega N/2}}{e^{-j\omega/2}}\frac{e^{j\omega N/2}-e^{-j\omega N/2}}{e^{j\omega/2}-e^{-j\omega/2}}$$

we can evaluate its amplitude using Euler relations in the form

$$\begin{aligned} \left|H(e^{j\omega})\right| &= \frac{1}{N}\left|\frac{cos(\omega N/2)+jsin(\omega N/2)-(cos(\omega N/2)-jsin(\omega N/2))}{cos(\omega/2)+jsin(\omega/2)-(cos(\omega/2)-jsin(\omega/2))}\right| = \\ &= \frac{1}{N}\left|\frac{sin(\omega N/2)}{sin(\omega/2)}\right| \end{aligned}$$

Amplitude frequency response given in Fig. 2.8 for $\omega \in \langle 0, \pi\rangle$ provides information about the system behaviour with respect to the signal frequency components.

Difference equations or state space representation provide possibilities for the time domain digital system application while the frequency response provides information about its behaviour with respect to its frequency components. Methods of parameter estimation enabling signal analysis or its processing to achieve prescribed system behaviour are studied in next sections.

3

Mathematical Background

Discrete signals are represented by sequences of values which implies the discrete system description by difference equations instead of differential equations for continuous signals. Resulting discrete system analysis and processing involves the following basic mathematical disciplines:

- *Z-transform* based upon the complex variable theory used for signal and system description
- theory of *difference equations* used for system representation
- *discrete Fourier transform* covering signal component analysis
- statistical methods including *stochastic processes* and *the least square method* fundamental for adaptive signal processing

Topics mentioned above are described in many special books and we shall summarize basic results only with notes to further detail references.

3.1 Z-transform and Signal Description

Z-transform is a mathematical tool closely connected with the theory of complex variable enabling compact signal and system description and giving possibility of its simple processing [23, p.76], [40, p.124], [42].

3.1.1 Definitions and Basic Properties

Definition 3.1 *The two-sided Z-transform of a sequence* $\{x(n)\}$ *is defined as*

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{3.1}$$

in the complex plane of variable z.

In case of a causal sequence having $x(n) = 0$ for $n < 0$ the transform is reduced to one-sided only with summation in Def. 3.1 for $n = 0, 1, \cdots, \infty$. In both cases the *region of convergence* covers the set of those z values for which the summation has a finite value.

Example 3.1 *Evaluate the Z-transform of the causal exponential sequence with its region of convergence.*

Solution: Using the unit step function it is possible to express the exponential sequence

$$x(n) = a^n u(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

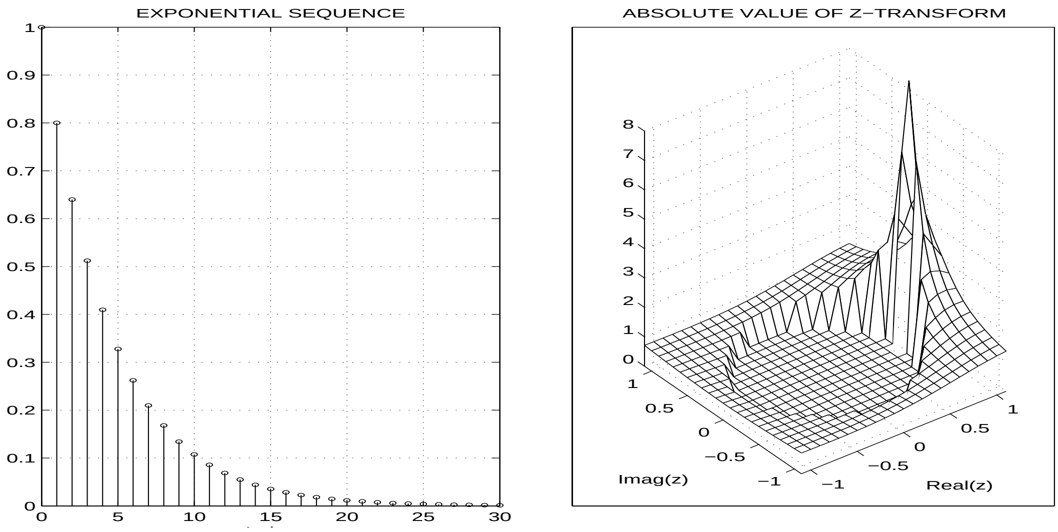

FIGURE 3.1. Exponential sequence $x(n) = a^n$ for $a = 0.8$ and absolute value of its Z-transform representation above the region of convergence for $|z| > |a|$ in the complex domain for $Re(z) \in \langle -1.1, 1.1 \rangle$ and $Im(z) \in \langle -1.1, 1.1 \rangle$

and to find

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

which represents the geometrical sequence having its value

$$X(z) = \frac{1}{1 - az^{-1}}$$

for quotient $|az^{-1}| < 1$ which implies $|z| > |a|$. Representation of original sequence and its Z-transform in the complex plane above the region of convergence is given in Fig. 3.1.

Using the Def. 3.1 it is possible to evaluate Z-transform of further sequences and the region of convergence as well. Some results are summarized in Tab. 3.1 presenting correspondence between original sequences $\{x(n)\}$ and their representation $X(z)$ in the complex plane. Advantages of such a transformation are obvious from the next section presenting possibilities of discrete system description and difference equation solution.

Example 3.2 *Evaluate the two-sided Z-transform of the exponential sequence*

$$x(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ b^n & \text{for } n < 0 \end{cases}$$

Solution: Using the Def. 3.1 it is possible to find

$$X(z) = \sum_{n=-1}^{-\infty} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=1}^{\infty} b^{-n} z^n + \sum_{n=0}^{\infty} a^n z^{-n}$$

In case that quotients of these geometrical sequences are in absolute values less than one which means that

$$|b^{-1}z| < 1 \qquad \text{and} \qquad |az^{-1}| < 1$$

or

$$|a| < |z| < |b|$$

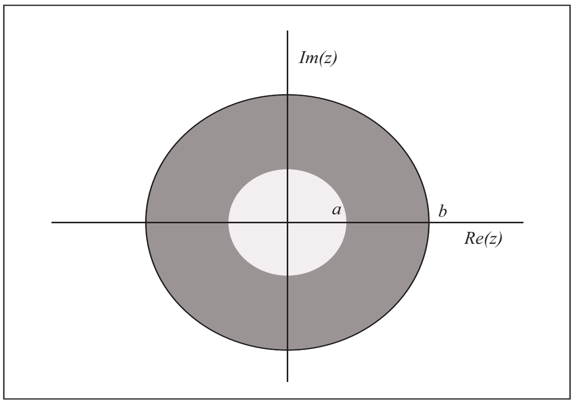

FIGURE 3.2. Region of convergence for the two-sided exponential sequence

it is possible to express the result in the form

$$X(z) = \frac{b^{-1}z}{1-b^{-1}z} + \frac{1}{1-az^{-1}} = -\frac{z}{z-b} + \frac{z}{z-a} = \frac{z(a-b)}{(z-a)(z-b)}$$

Region of convergence is given in Fig. 3.2. It is obvious that according to values $|a|$ and $|b|$ it can be empty as well.

Fundamental properties of the Z-transform can be stated in the following form

1. Linearity
$$Z[ax_1(n) + bx_2(n)] = aZ[x_1(n)] + bZ[x_2(n)] \tag{3.2}$$

2. Translation
$$Z[x(n)] = X(z) \quad \Rightarrow \quad Z[x(n-m)] = z^{-m}X(z) \tag{3.3}$$

3. Convolution in time domain
$$Z[\sum_{k=-\infty}^{\infty} x(k)y(n-k)] = Z[x(n)] \cdot Z[y(n)] \tag{3.4}$$

4. Initial value theorem (for causal sequences)
$$x(0) = \lim_{z\to\infty} X(z) \tag{3.5}$$

Proofs of these properties result from the Def. 3.1.

Sequence	*Definition*	*Z-transform*	*Region of convergence*
Unit sample	$d(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$	1	all z
Unit step	$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$	$\frac{z}{z-1}$	$\|z\| > 1$
Exponential	$x(n) = a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
Harmonic	$x(n) = \sin(2\pi fn)$	$\frac{z\sin(2\pi f)}{z^2 - 2z\cos(2\pi f) + 1}$	$\|z\| > 1$
	$x(n) = \cos(2\pi fn)$	$\frac{z^2 - z\cos(2\pi f)}{z^2 - 2z\cos(2\pi f) + 1}$	$\|z\| > 1$

TABLE 3.1. Basic sequences and their Z-transform

3.1.2 Inverse Z-transform

Polynomial $X(z)$ defined by Eq. (3.1) is determined by the complete sequence $x(n)$ and it enables its reconstruction as well confirming in this way the equivalence between the sequence definition in time and complex domains. This process of the inverse Z-transform can be performed in several ways.

The application of COMPLEX INVERSION INTEGRAL is based upon the complex variable theory [23, p.76], [42], [12, p.770] enabling the derivation of the Z-transform inversion formula in the form

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) \frac{z^n}{z} dz \tag{3.6}$$

with C representing the closed contour laying inside the region of convergence.

As the Z-transform definition usually results in the rational representation of $X(z)$ the PARTIAL FRACTION EXPANSION METHOD may be used to express the original function as a sum of simple fractions in the following way.

1. Evaluation of poles $p_0, p_1, ..., p_N$ of

$$X(z) = \frac{b_0 z^N + \cdots + b_N}{a_0 z^N + \cdots + a_N} \tag{3.7}$$

   with some possible zero coefficients and unequal order of numerator and denominator polynomials.

2. Partial fraction expansion of function $X(z)/z$ based upon the knowledge that z appears in the numerators of functions $X(z)$

$$\frac{X(z)}{z} = \frac{c_0}{z - p_0} + \frac{c_1}{z - p_1} + \cdots + \frac{c_N}{z - p_N} + (k_1 + k_2 z + \cdots) \tag{3.8}$$

   and evaluation of coefficients $c_0, c_1, \cdots, c_N$. Direct terms with coefficients $k_1, k_2, \cdots$ appear for non-proper fractions only. As complex poles are in complex conjugate pairs they can be combined into second order terms before further processing.

3. Using expression

$$X(z) = \frac{c_0 z}{z - p_0} + \cdots + \frac{c_N z}{z - p_N} + (k_1 z + k_2 z^2 + \cdots) \tag{3.9}$$

   with possible second order terms we can use the Z-transform table in connection with the knowledge of basic properties and the region of convergence to find the original sequence.

Example 3.3 *Evaluate the causal sequence $x(n)$ having its Z-transform in the form*

$$X(z) = \frac{0.3z}{z^2 - 0.7z + 0.1}$$

Solution: As

$$\frac{X(z)}{z} = \frac{0.3}{(z - 0.2)(z - 0.5)} = \frac{c_0}{z - 0.2} + \frac{c_1}{z - 0.5}$$

it is possible to find coefficients c_0, c_1 from the following equation

$$0.3 = c_0(z - 0.5) + c_1(z - 0.2)$$

The previous equation must be valid for all z which implies that

$$\begin{aligned} 0.3 &= -0.5c_0 - 0.2c_1 \\ 0 &= c_0 + c_1 \end{aligned}$$

giving solution $c_0 = -1, c_1 = 1$. As further

$$X(z) = -\frac{z}{z - 0.2} + \frac{z}{z - 0.5}$$

the Tab. 3.1 enables evaluation of

$$x(n) = -(0.2)^n u(n) + (0.5)^n u(n)$$

Computer processing of the partial fraction expansion method may be based upon the procedure presented in Alg. 3.1. For realization in other languages than MATLAB compact functions presented here must be realized in other way and in most cases by special subroutines.

Algorithm 3.1 *Evaluation of the partial fraction expansion for the rational function*

$$X(z) = \frac{[b(0) \cdots b(N)][z^N \cdots 1]'}{[a(0) \cdots a(N)][z^N \cdots 1]'} \tag{3.10}$$

- definition of vectors $\mathbf{b}$ and $\mathbf{a}$ of the rational function
- evaluation of vectors $\mathbf{c}, \mathbf{p}$ and $\mathbf{k}$ of expansion

  $$X(z) = \frac{c_0 z}{z - p_0} + \cdots + \frac{c_N z}{z - p_N} + (k_1 z + k_2 z^2 + \cdots)$$

  by function

  $[\mathbf{c}, \mathbf{p}, \mathbf{k}] = \text{residue } (\mathbf{b}, \mathbf{a})$
- as the inverse procedure can be realized by function

  $[\mathbf{b}, \mathbf{a}] = \text{residue } (\mathbf{c}, \mathbf{p}, \mathbf{k})$

  it can be used for connection of terms with complex conjugate poles to form second order terms of partial fraction expansion with real coefficients only to enable the following use of the Z-transform tables

Division method provides possibility of evaluation of individual members of the original sequence based upon the knowledge of its Z-transform $X(z)$ and the region of convergence. In case of causal sequences the expansion of $X(z)$ can be restricted to non-positive powers of z in the form

$$X(z) = x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \cdots \tag{3.11}$$

with coefficients $x(0), x(1), \cdots$ defining the desired sequence.

Example 3.4 *Evaluate the inverse Z-transform of*

$$X(z) = \frac{0.3z}{z^2 - 0.7z + 0.1}$$

for region of convergence defined by $|z| > 0.5$.

Solution: Dividing the rational function $X(z)$ we obtain

$$\begin{array}{l}
+0.3z \qquad\qquad\qquad\qquad : (z^2 - 0.7z + 0.1) = 0.3z^{-1} + 0.21z^{-2} + 0.117z^{-3} + \cdots \\
\underline{\pm 0.3z \mp 0.21 \pm 0.030z^{-1}} \\
\qquad + 0.21 - 0.030z^{-1} \\
\qquad \underline{\pm 0.21 \mp 0.147z^{-1} \pm 0.0210z^{-2}} \\
\qquad\qquad + 0.117z^{-1} - 0.0210z^{-2} \\
\qquad\qquad \underline{\pm 0.117z^{-1} \mp 0.0890z^{-2} \pm 0.0117z^{-3}} \\
\qquad\qquad\qquad\qquad + 0.0609z^{-2} - 0.0112z^{-3}
\end{array}$$

The desired sequence has the following values:

$$\begin{aligned} x(n) &= 0 \qquad \text{for } n \leq 0 \\ x(1) &= 0.3 \\ x(2) &= 0.21 \\ &\cdots \end{aligned}$$

Results of the example evaluated by Alg. 3.2 are presented in Fig. 3.3.

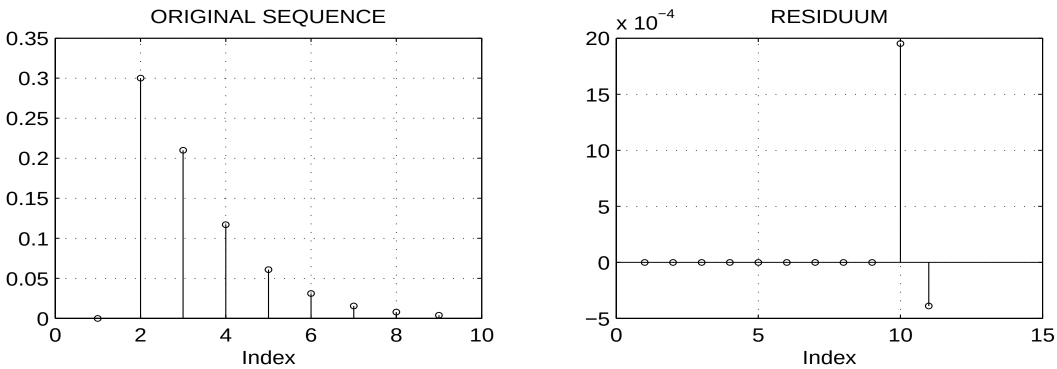

FIGURE 3.3. Sequence {x(n)} for $n = 1, \cdots, L+1$ evaluated as the inverse Z-transform to $X(z) = (0.3z)/(z^2 - 0.7z + 0.1)$ for $L = 8$ and the residuum sequence

3.2 Difference Equations and System Modelling

The linear shift invariant discrete system is an essential mathematical structure for the approximation of most continuous real systems. It can be used for their modelling, analysis and signal processing as well.

3.2.1 System Representation

The description of the linear shift invariant system can be given by the *difference equation* in the general form

$$y(n) + \sum_{k=1}^{N} a(k) y(n-k) = \sum_{k=0}^{N} b(k) x(n-k) \tag{3.13}$$

with some possible zero coefficients. This time domain representation can be further modified to enable more convenient ways of digital signal processing.

The *discrete transfer function* (system function) can be derived from the Z-transform of the difference Eq. (3.13) resulting in relation

$$Z[y(n)] + \sum_{k=1}^{N} a(k) Z[y(n-k)] = \sum_{k=0}^{N} b(k) Z[x(n-k)]$$

Using the translation property of the Z-transform we obtain

$$Y(z) + \sum_{k=1}^{N} a(k) z^{-k} Y(z) = \sum_{k=0}^{N} b(k) z^{-k} X(z)$$

and the transfer function in the form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{N} a(k) z^{-k}} \tag{3.14}$$

This representation enables simple evaluation of the system output in the following steps

- description of the input sequence $\{x(n)\}$ in the form of its Z-transform X(z)
- application of the transfer function $H(z)$ for evaluation of the output sequence Z-transform
$$Y(z) = H(z)X(z) \tag{3.15}$$
- evaluation of the output sequence $\{y(n)\}$ by the inverse Z-transform of Y(z)

Any method of the inverse Z-transform can be used in this stage.

Algorithm 3.2 *Polynomial division for the evaluation of the sequence* $\{x(n)\}$ *representing the inverse Z-transform in the form*

$$\begin{aligned} X(z) &= \frac{[b(0) \cdots b(N)][z^N \cdots 1]'}{[a(0) \cdots a(N)][z^N \cdots 1]'} \\ &= [x(0) \cdots x(L)][z^0 z^{-1} \cdots z^{-L}]' + \frac{[r(L+1) \cdots r(L+N)][z^{N-L-1} \cdots z^{-L}]'}{[a(0) \cdots a(N)][z^N \cdots 1]'} \end{aligned} \tag{3.12}$$

- definition of vectors $\mathbf{d}$ and $\mathbf{a}$ where $\mathbf{d} = [\mathbf{b}, zeros(1, L)]$ represents a new numerator vector after the multiplication of the whole expression by z^L to enable expansion of the newly defined non-proper fraction to vector $\mathbf{x}$ connected with non-negative powers of $\mathbf{z}$ and the remainder vector $\mathbf{r}$.
- evaluation of values $[x(0), \cdots, x(L)]$ by function
  $[\mathbf{x}, \mathbf{r}] = \text{decom}\ (\mathbf{d}, \mathbf{a})$
- possible graphic representation of the evaluated sequence by function
  $\text{plot}(\mathbf{x})$

The *unit sample response* represents another possibility of system description. As $X(z) = 1$ for such a sequence the Z-transform of system output can be evaluated using (3.15) resulting in $Y(z) = H(z)$ which implies that the inverse Z-transform of $H(z)$ stands for the unit sample response $\{h(n)\}$. The transfer function for causal system is then defined by relation

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{3.16}$$

and it is equivalent to that defined by Eq. (3.14). As $H(z) = Y(z)/X(z)$ it is obvious that any input sequence $\{x(n)\}$ implies system output $\{y(n)\}$ in the form

$$y(n) = \sum_{k=0}^{\infty} h(k) x(n-k) = h(n) * x(n) \tag{3.17}$$

referred as convolution of sequences $\{h(n)\}$ and $\{x(n)\}$.

The system *frequency response* $H(e^{j\omega})$ can be evaluated after the application of the input sequence $x(n) = e^{j\omega n}$ to the system described by difference Eq. (3.13) or (3.17). Using Eq. (3.17) it is obvious that

$$y(n) = \sum_{k=0}^{\infty} h(k) e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=0}^{\infty} h(k) e^{-j\omega k} = x(n) \sum_{k=0}^{\infty} h(k) e^{-j\omega k} \tag{3.18}$$

Comparing Eq. (3.16) and (3.18) it is possible to express the frequency response

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} \tag{3.19}$$

having its magnitude and phase part.

Results described above imply the basic role of the transfer function $H(z)$ in form of Eq. (3.14) or (3.16) enabling the difference equation or frequency response evaluation.

Example 3.5 *Use the transfer function*

$$H(z) = 0.2 \frac{z+1}{z^2 - z + 0.5} \tag{3.20}$$

of a causal system to evaluate its difference equation, unit sample response and frequency response.

Solution:

- As
$$H(z) = \frac{Y(z)}{X(z)} = 0.2 \frac{z+1}{z^2 - z + 0.5} = 0.2 \frac{z^{-1} + z^{-2}}{1 - z^{-1} + 0.5 z^{-2}}$$
we can find after the cross multiplication that
$$Y(z)(1 - z^{-1} + 0.5 z^{-2}) = 0.2 X(z)(z^{-1} + z^{-2})$$
which after the inverse Z-transform results in the difference equation
$$y(n) - y(n-1) + 0.5 y(n-2) = 0.2(x(n-1) + x(n-2))$$

- One of possibilities how to evaluate the unit sample response is to use the transfer function to obtain
$$Y(z) = H(z)\, X(z)$$
Taking into account the unit sample Z-transform $X(z) = 1$ it is possible to use the division method to evaluate separate terms of $\{h(n)\}$. As

$$
\begin{array}{l}
(\ +0.2z+0.2) \qquad\qquad : (z^2-z+0.5) = 0.2z^{-1}+0.4z^{-2}+0.3z^{-3} \\
\underline{-\,0.2z \mp 0.2 \pm 0.1z^{-1}} \\
\qquad +0.4-0.1z^{-1} \\
\qquad \underline{-\,0.4 \mp 0.4z^{-1} \pm 0.2z^{-2}} \\
\qquad\qquad +0.3z^{-1}-0.2z^{-2} \\
\qquad\qquad \underline{-\,0.3z^{-1} \mp 0.3z^{-2} \pm 0.15z^{-3}} \\
\qquad\qquad\qquad +0.1z^{-2}-0.15z^{-3}
\end{array}
$$

the resulting sequence $\{h(n)\}_{n=0}^{\infty}$ has values $\{0, 0.2, 0.4, 0.3, \cdots\}$.

- The frequency response can be evaluated using Eq. (3.19) in form

$$H(e^{j\omega}) = H(z)\mid_{z=e^{j\omega}} = 0.2\frac{e^{j\omega}+1}{e^{2j\omega}-e^{j\omega}+0.5}$$

After application of Euler relations it is possible to write

$$H(e^{j\omega}) = 0.2\frac{1+\cos(\omega)+j\ sin(\omega)}{(\cos(2\omega)-\cos(\omega)+0.5)+j(\sin(2\omega)-\sin(\omega))}$$

The magnitude and phase of this frequency response for $\omega \in \langle 0, \pi\rangle$ is presented in Fig. 3.4 together with the sketch of the given transfer function representation in the complex plane showing its poles and values on the unit circle for $z = e^{j\omega}$.

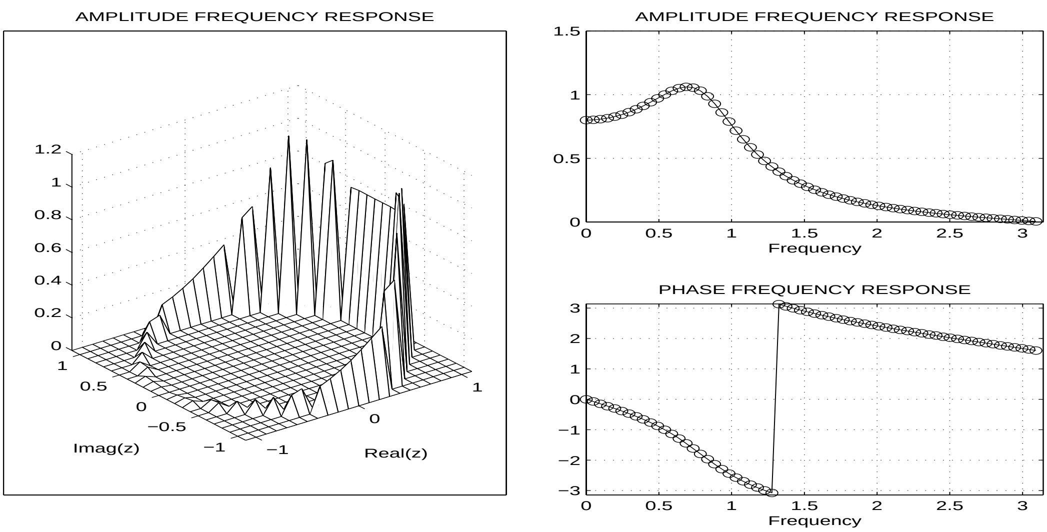

FIGURE 3.4. Magnitude and phase frequency response of the discrete system with the transfer function $H(z) = 0.2(z+1)/(z^2-z+0.5)$ and its sketch in the complex plane.

Frequency response provides a very important information concerning the system behaviour with respect to the input signal frequency components. Computer processing of the system response and frequency response based upon the knowledge of the discrete transfer function can be summarized in Algorithm 3.3 and 3.4.

3.2.2 Linear Constant Coefficients Difference Equations

The *classical solution* of difference equations is very close to methods of solution of differential equations and it involves the estimation of the particular and homogenous solution as well [36, p.16].

Algorithm 3.3 *System response evaluation for the transfer function*

$$H(z) = \frac{[b(0), b(1) \cdots b(N)][1, z^{-1} \cdots z^{-N}]'}{1 + [a(1) \cdots a(N)][z^{-1} \cdots z^{-N}]'} \tag{3.21}$$

to the input sequence

$$\mathbf{x} = [x(0), x(1) \cdots]$$

- definition of vectors $\mathbf{b}, \mathbf{a}$ and $\mathbf{x}$.
- system output evaluation by function
  $\mathbf{y}$ = filter($\mathbf{b}, \mathbf{a}, \mathbf{x}$)
- possible graphic output of the original and evaluated sequence (with two pictures on the screen)
  clg; subplot(211);
  plot($\mathbf{x}$); plot($\mathbf{y}$)

The *Z-transform method* provides another possibility of a very simple way for solution of the equation

$$y(n) + \sum_{k=1}^{N} a(k)y(n-k) = f(n) \tag{3.23}$$

with a given set of initial conditions $\{y(-1), y(-2), \cdots, y(-N)\}$. The solution consists in principle of the following steps

- Z-transform application which transforms the difference equation into an algebraic equation
- evaluation of $Y(z)$ standing for the Z-transform of the solution
- inverse Z-transform application for evaluation of $\{y(n)\}$.

Algorithm 3.4 *Frequency response evaluation of system defined by its transfer function*

$$H(z) = \frac{[b(0), b(1) \cdots b(N)][1, z^{-1} \cdots z^{-N}]'}{1 + [a(1) \cdots a(N)][z^{-1} \cdots z^{-N}]'} \tag{3.22}$$

- definition of vectors $\mathbf{b}$ and $\mathbf{a}$.
- frequency response evaluation by function
  $[\mathbf{h}, \mathbf{w}]$ =freqz($\mathbf{b}, \mathbf{a}, n$)
  in n points between 0 and π defined in vector $\mathbf{w}$ with result in vector $\mathbf{h}$.
- possible separate plots of magnitude and phase of the frequency response (with two pictures on the screen)
  clg; subplot(211);
  plot($\mathbf{w}$, abs($\mathbf{h}$)); plot($\mathbf{w}$, angle($\mathbf{h}$))

The Z-transform of real causal sequence $\{y(n)u(n)\}$ can be defined as $Y(z)$. To obtain the Z-transform of the delayed truncated sequence we can evaluate

$$\begin{aligned} Z[y(n-k)u(n)] &= \sum_{n=-\infty}^{\infty} y(n-k)u(n)z^{-n} = \sum_{n=0}^{\infty} y(n-k)z^{-n} = \sum_{m=-k}^{\infty} y(m)z^{-(m+k)} = \\ &= \sum_{m=-k}^{-1} y(m)z^{-(m+k)} + z^{-k}\sum_{k=0}^{\infty} y(m)z^{-m} \end{aligned}$$

which implies that

$$Z[y(n-k)u(n)] = \sum_{m=-k}^{-1} y(m)z^{-(m+k)} + z^{-k}Y(z)$$

with the first term enabling to apply initial conditions of Eq. (3.23).

Example 3.6 *Evaluate the solution of the following linear constant difference equation*

$$y(n) - 0.5y(n-1) = 0.25^n$$

for $y(-1) = 1$.

Solution: After the Z-transform we shall receive

$$Y(z) - 0.5(y(-1) + z^{-1}Y(z)) = \frac{1}{1-0.25z^{-1}}$$

which implies that

$$Y(z) = \frac{\frac{1}{1-0.25z^{-1}} + 0.5}{1-0.5z^{-1}} = \frac{1.5 - 0.125z^{-1}}{(1-0.5z^{-1})(1-0.25z^{-1})} = \frac{1.5z^2 - 0.125z}{(z-0.5)(z-0.25)}$$

Using the partial fraction expansion method we obtain

$$Y(z) = \frac{2.5z}{z-0.5} - \frac{z}{z-0.25}$$

which implies the solution in the following form

$$y(n) = 2.5(0.5)^n - (0.25)^n$$

3.3 Discrete Fourier Transform and Signal Decomposition

Discrete signals and systems described by difference equations can be represented by Z-transform enabling their simple analysis and further manipulation. Another way of signal processing is based upon its decomposition into a linear combination of basis functions [23, p.257]. In linear time invariant system various methods can be then applied separately to signal components and results composed again. This method is essential in many engineering applications enabling signal analysis, filtering of signal parts, adaptive signal processing etc.

Physical bases of many signals enable their harmonic decomposition which implies that the weighted sum of complex exponentials is used very often. Therefore the discrete Fourier transform based upon the Fourier series for periodic signals is an essential mathematical tool for the theoretical analysis of many digital signal processing methods and it enables their implementation using an efficient algorithm of the fast Fourier transform [7] as well.

3.3.1 Definition and Basic Properties

To explain the definition of the discrete Fourier transform we can start with representation of periodic discrete-time signal $\{x(n)\}$ with period N by the weighted sum of complex exponentials in the form

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{jk\frac{2\pi}{N}n} \tag{3.24}$$

for $n = 0, 1, \cdots, N-1$. This expression is in close connection with Fourier series applied to continuous signals for the infinitive sum reduced to the finite sum of N terms only caused by N distinct exponentials for frequencies

$$\omega_k = k\frac{2\pi}{N}, \qquad k = 0, 1, \cdots, N-1.$$

The multiplying constant $1/N$ in Eq. (3.24) has no substantial effect in this stage. To evaluate terms $X(k)$ we can multiply both sides of Eq. (3.24) by $e^{-jl(2\pi/N)n}$ and to sum over $n = 0, 1, \cdots, N-1$ to obtain after the interchange of the summation order [30, p.88]

$$\sum_{n=0}^{N-1} x(n) e^{-jl\frac{2\pi}{N}n} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} e^{j(k-l)\frac{2\pi}{N}n}$$

Relation

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{j(k-l)\frac{2\pi}{N}n} = \begin{cases} 1 & \text{for } k-l = mN \\ 0 & \text{for } k-l \neq mN \end{cases}$$

implies that

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-jk\frac{2\pi}{N}n} \tag{3.25}$$

This result can be also applied to finite sequences of N samples in case that we define the periodic sequence based upon the periodic extension of original values. It is often assumed that the nonzero period is for $n \in \langle 0, N-1 \rangle$. The discrete Fourier transform is then defined by the following relations.

Definition 3.2 *Let us assume the finite sequence $\{x(n)\}$ for $n = 0, 1, \cdots, N-1$. Its discrete Fourier transform is then defined by relation*

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-jk\frac{2\pi}{N}n} \tag{3.26}$$

for $k = 0, 1, \cdots, N-1$.

The inverse transform can be evaluated by relation

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{jk\frac{2\pi}{N}n} \tag{3.27}$$

for $n = 0, 1, \cdots, N-1$ and discrete frequencies $\omega_k = k2\pi/N$. Relation (3.26) defines in fact coefficients of Eq. (3.27) related to separate frequency components.

Example 3.7 *Evaluate the discrete Fourier transform of a given sequence*

$$x(n) = \begin{cases} 1 & \textit{for } 0 \leq n \leq 2 \\ 0 & \textit{for } 3 \leq n \leq 8 \end{cases} \tag{3.28}$$

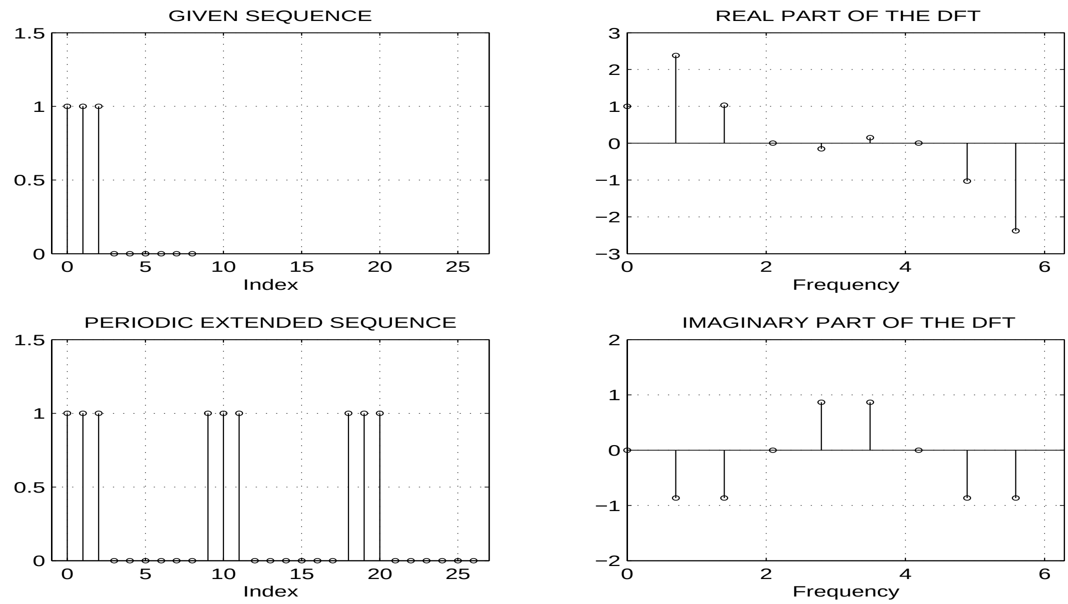

FIGURE 3.5. Discrete Fourier transform of a given sequence

Solution: Using the Def. 3.2 we can write

$$X(k) = \sum_{n=0}^{9} x(n) e^{-jk(2\pi/9)n} = \sum_{n=0}^{2} e^{-jk(2\pi/9)n}$$

which is a geometrical sequence implying

$$X(k) = \frac{1 - e^{-jk(2\pi/9)3}}{1 - e^{jk(2\pi/9)}} = \frac{e^{-jk(\pi/3)}}{e^{-jk(\pi/9)}} \frac{e^{jk(\pi/3)} - e^{-jk(\pi/3)}}{e^{jk(\pi/9)} - e^{-jk(\pi/9)}}$$

Using Euler relations we shall receive

$$X(k) = e^{-jk(2\pi/9)} \frac{\sin(k(\pi/3))}{\sin(k(\pi/9))} \tag{3.29}$$

Graphical representation of results in presented in Fig. 3.5.

Computer processing of the discrete Fourier transform can be based upon Algorithm 3.5 using simple MATLAB notation.

Discrete Fourier transform is closely related to the *Z-transform* which implies similar properties of both transforms as well. As the finite length sequence $\{x(n)\}$ for $n = 0, 1, \cdots, N-1$ has its Z-transform according to the definition in the form

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \tag{3.30}$$

the comparison of Eqs. (3.30) and (3.26) results in relation

$$X(k) = X(z) \mid_{z=e^{jk2\pi/N}} \tag{3.31}$$

for $k = 0, 1, \cdots, N-1$. This result implies that the discrete Fourier transform represents equidistant values of $X(z)$ on the unit circle in the complex plane [30, p.90].

Example 3.8 *Evaluate the discrete Fourier transform of the exponential sequence*

$$x(n) = \begin{cases} a^n & \textit{for } n = 0, 1, \cdots, N-1 \\ 0 & \textit{for } n < 0 \textit{ and } n \geq N \end{cases}$$

Algorithm 3.5 *Evaluation of the direct and inverse discrete Fourier transform of sequence* $\{x(n)\}$, $n = 0, 1, \cdots, N-1$.

- definition of vector $\mathbf{x} = [x(0), \cdots, x(N-1)]$
- discrete Fourier transform evaluation

  $\mathbf{X} = \text{fft}(\mathbf{x})$
- graphic separate representation of the real and imaginary part

  subplot(211)
  plot((0 : $N-1$)./N, real($\mathbf{X}$))
  plot((0 : $N-1$)./N, imag($\mathbf{X}$))
- inverse discrete Fourier transform evaluation

  $\mathbf{y} = \text{ifft}(\mathbf{X})$

Solution: As

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{z^N - a^N}{z^N - az^{N-1}}$$

it is possible to evaluate

$$X(k) = X(z)\mid_{z=e^{jk2\pi/N}} = \frac{e^{jk2\pi} - a^N}{e^{jk2\pi} - ae^{jk2\pi(N-1)/N}} = \frac{1-a^N}{1-ae^{jk2\pi(N-1)/N}}$$

for $k = 0, 1, \cdots, N-1$. The geometrical view presenting real and imaginary part of the discrete Fourier transform and its absolute value as a special case of the Z-transform on the unit circle in the complex plane for $N = 24$ discrete frequencies is given in Fig. 3.6. Separate plots of real and imaginary parts of the discrete Fourier transform are presented in Fig. 3.7 in connection with the complex plane interpretation again.

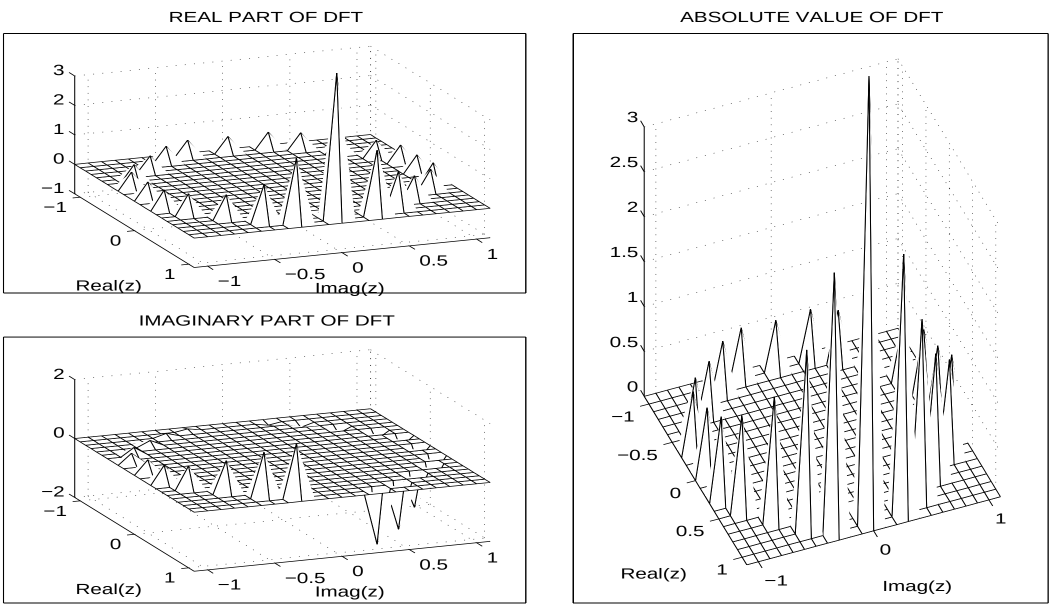

FIGURE 3.6. The discrete Fourier transform of exponential sequence related to its Z-transform in the complex plane

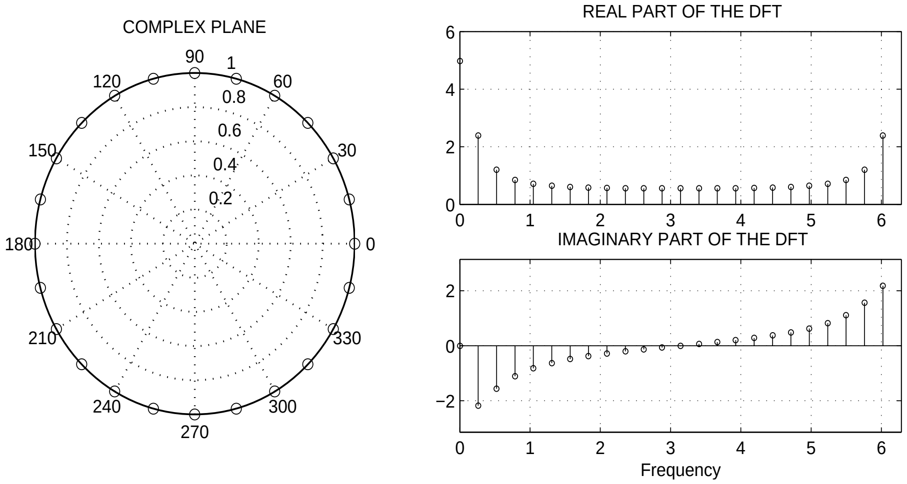

FIGURE 3.7. The real and imaginary part of the DFT of the exponential sequence in connection with the complex plane interpretation for $\omega_k = k2\pi/N, k = 0, 1, \cdots, N-1$ (for $N = 24$)

The graphic interpretation of the discrete Fourier transform given in the previous example enables better understanding of the frequency axis description given in Fig. 3.8 and it presents its *symmetry properties* as well. As terms $e^{jk2\pi/N}$ and $e^{j(N-k)2\pi/N}$ for $k = 0, 1, \cdots, N$ are complex conjugates the Eq. (3.26) implies that $X(k)$ and $X(N-k)$ are for real values of $\{x(n)\}$ in the same relation [39, p.252] which means that

- $real(X(k))$ is an even function in such a sense that $real(X(k)) = real(X(N-k))$
- $imag(X(k)$ is an odd function in such a sense that $imag(X(k)) = -imag(X(N-k))$
- $abs(X(k))$ is an even function

The definition of even and odd function is based upon the periodic extension of the analysed values. It is obvious that owing to this properties it is sufficient to evaluate $X(k)$ for $k = 0, 1, \cdots, N/2$ only.

Further *fundamental properties* of the discrete Fourier transform of a sequence $\{x(n)\}, n = 0, 1, \cdots, N-1$ can be stated in the following form [39, p.258].

1. Linearity
$$DFT[a_1x_1(n) + a_2x_2(n)] = a_1DFT[x_1(n)] + a_2DFT[x_2(n)] \qquad (3.32)$$
2. Translation
$$DFT[x(n] = X(k) \quad \Rightarrow \quad DFT[x(n-m)] = e^{-jkm\frac{2\pi}{N}}X(k) \qquad (3.33)$$
3. Convolution in time domain
$$DFT[\sum_{k=0}^{N-1} x(k)y(n-k)] = DFT[x(n)] \cdot DFT[y(n)] \qquad (3.34)$$

Proofs of these properties result from the Def. 3.2.

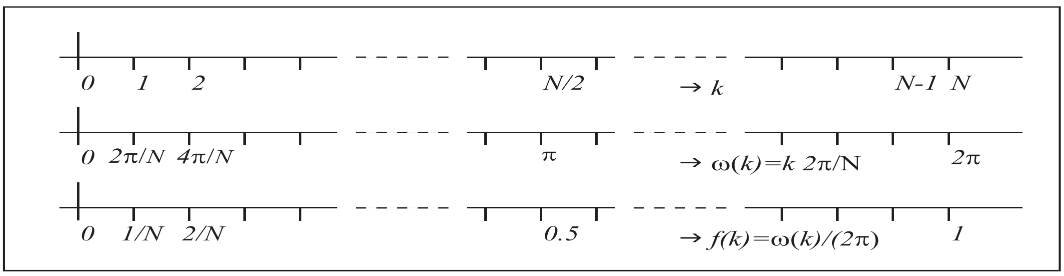

FIGURE 3.8. Frequency axis interpretation

3.3.2 Fast Fourier transform

Definition of the discrete Fourier transform (DFT) enables the estimation of basic numerical calculations of this method reaching the order of N^2 for complex multiplications and additions. The fast Fourier transform (FFT) algorithm reduces the required number of arithmetic operations to the order of $(N/2)log_2(N)$ which for $N = 512$ means the approximate reduction to 1% of the original value connected with time requirements as well.

Let us assume sequence $x(n)_{n=0}^{N-1}$ with its length N being a power of 2 and its discrete Fourier transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-jk\frac{2\pi}{N}n}.$$

The first stage of the algorithm [23, p.272] is based upon its breaking into the sum of even-indexed and odd-indexed data $\{x(n)\}$ to define the following expression

$$X(k) = \sum_{n=0}^{N/2-1} x(2n) e^{-jk\frac{2\pi}{N}2n} + \sum_{n=0}^{N/2-1} x(2n+1) e^{-jk\frac{2\pi}{N}(2n+1)} \tag{3.35}$$

which results in

$$X(k) = \sum_{n=0}^{N/2-1} x(2n) e^{-jk\frac{2\pi}{N/2}n} + e^{-jk\frac{2\pi}{N}} \sum_{n=0}^{N/2-1} x(2n+1) e^{-jk\frac{2\pi}{N/2}n} \tag{3.36}$$

It can be seen that computation of the DFT of length N has been reduced to the computation of two transforms of length $N/2$ and an additional $N/2$ complex multiplications for the complex exponential outside the second summation considering $k = 0, 1, \cdots, N/2 - 1$. It would appear at first sight that it is necessary to evaluate Eq. (3.36) for $k = 0, 1, \cdots, N-1$. However it is not the truth as may be seen by considering result for indices $k+N/2$ having the following form

$$X(k + \frac{N}{2}) = \sum_{n=0}^{N/2-1} x(2n) e^{-j(k+\frac{N}{2})\frac{2\pi}{N/2}n} + e^{-j(k+\frac{N}{2})\frac{2\pi}{N}} \sum_{n=0}^{N/2-1} x(2n+1) e^{-j(k+\frac{N}{2})\frac{2\pi}{N/2}n}$$

which owing to the periodicity results in

$$X(k + \frac{N}{2}) = \sum_{n=0}^{N/2-1} x(2n) e^{-jk\frac{2\pi}{N/2}n} - e^{-jk\frac{2\pi}{N}} \sum_{n=0}^{N/2-1} x(2n+1) e^{-jk\frac{2\pi}{N/2}n} \tag{3.37}$$

Comparing Eqs. (3.36) and (3.37) it is obvious that the only difference is in the sign between the two summations. Thus it is necessary to evaluate Eq. (3.36) for $k=0, 1,\cdots, N/2$–1 only storing the result of the two summations separately for each k. The values of $X(k)$ and $X(k + N/2)$ can then be evaluated as the sum and difference of the two summations as indicated by Eqs. (3.36) and (3.37). Thus the computational load for an N-point DFT has been reduced from N^2 operations to $2(N/2)^2 + N/2$. The flow chart for incorporating this decomposition into the computation of an $N = 8$ point DFT is presented in Fig. 3.9.

The same process can be carried out on each of the $N/2$ points of the transform to reduce further the computations. The flow chart for incorporating this extra stage of decomposition into the computation of the $N = 8$ point DFT is shown in Fig. 3.10. It can be seen that if $N = 2^M$ then the process can be repeated M times to reduce the

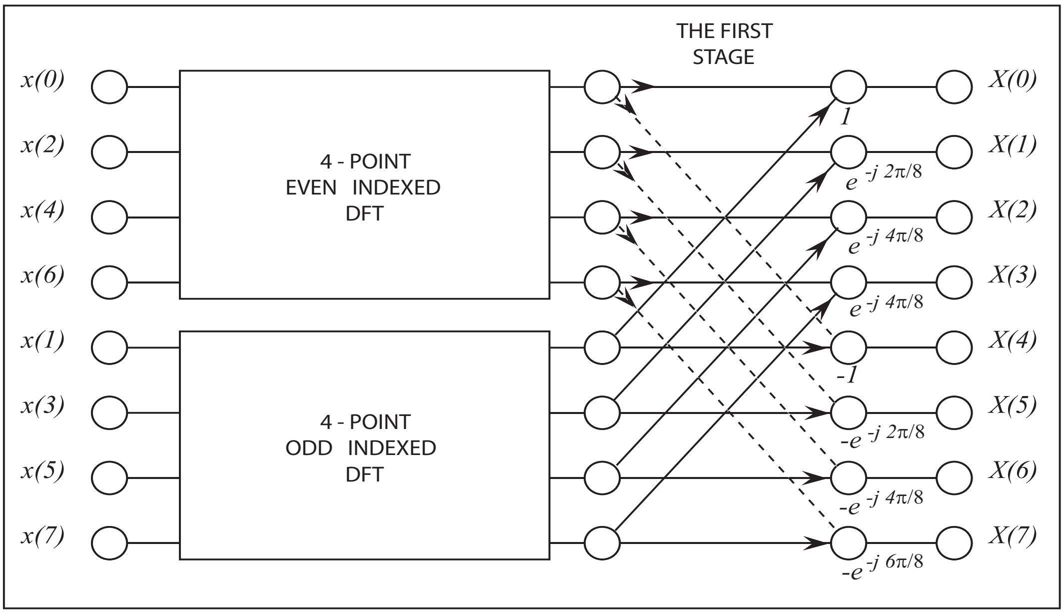

FIGURE 3.9. The first stage of the fast Fourier transform decomposition for $N = 8$

computation to that of evaluating N single point DFTs. The final flow chart for $N = 8$ presented in Fig. 3.10 is based upon the "butterfly" structure of the $N = 2$ point DFT of a sequence $\{s(0), s(1)\}$ evaluating

$$S(0) = s(0) + s(1)e^{-j0\frac{2\pi}{2}} = s(0) + s(1) \quad (3.38)$$

$$S(1) = s(0) + s(1)e^{-j1\frac{2\pi}{2}} = s(0) - s(1) \quad (3.39)$$

It is obvious that for the algorithm presented above it is necessary to shuffle the order of the input data. This data shuffle is usually termed "bit reversal" for reasoning that are clear if the indices of the shuffle data are written in binary as shown in Tab. 3.2.

It can be seen that the process reduces at each stage the computation by half but introduces an extra $N/2$ multiplications to account for the complex exponential term outside the second summation term in the reduction. Thus for the condition of $N = 2^M$ the process can be repeated M times to reduce the computation to that of evaluating N single point DFTs which require no computation. However at each of the M stages of reduction an extra $N/2$ multiplications is introduced so that the total number of arithmetic operations require to evaluate an N-point DFT is $(N/2)log_2(N)$.

binary	000	001	010	011	100	101	110	111
bit reversal	000	100	010	110	001	101	011	111
decimal	0	4	2	6	1	5	3	7

TABLE 3.2. Bit reversal used in the algorithm of the fast Fourier transform.

The FFT algorithm has a further significant advantage over the direct evaluation of the DFT expression in the fact that computation can be performed *on-place.* This has been illustrated in the final flow chart where it can be seen that after two data values have been processed by the *butterfly structure* those data are not required again in the computation and they may be replaced in the computer store with the values at the output of the butterfly structure. Computational algorithm of the fast Fourier transform is used in Algorithm 3.5 presented before.

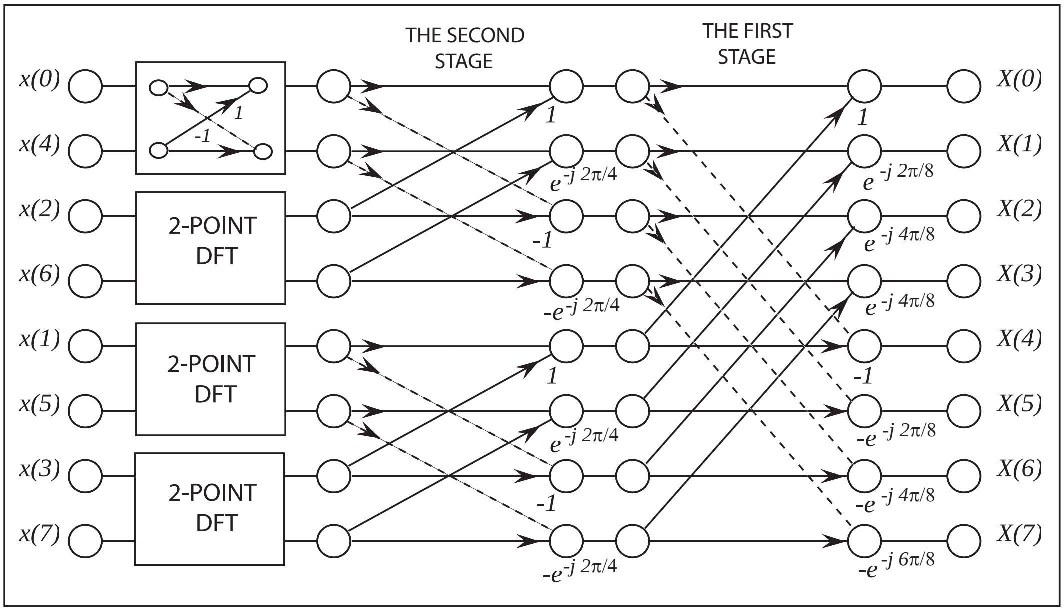

FIGURE 3.10. The first and second stage of the fast Fourier transform decomposition for $N = 8$

It is obvious from the definition of the direct and inverse discrete Fourier transform that the fast algorithm applied to obtain transformed values can be used with slight modifications in both directions.

3.3.3 Signal Decomposition and Reconstruction

Problem of the sampling rate estimation can be simply studied in connection with one harmonic component of the continuous-time periodic signal in the form

$$x_a(t) = cos(\Omega_a t) \tag{3.40}$$

sampled with the *sampling period* T_s to define sequence

$$x(n) = cos(\Omega_a n T_s) \tag{3.41}$$

for $n = 0, 1, \cdots$. Instead of the *analogue* frequency Ω_a [rad/s] we can introduce normalized *digital* frequency $\omega_d = \Omega_a T_s$ [rad] implying

$$x(n) = cos(\omega_d n) \tag{3.42}$$

Let us restrict our attention now to the finite length sequence having N samples and let us apply direct and inverse Fourier transform for its decomposition and reconstruction.

The SIGNAL DECOMPOSITION involves the application of the DFT definition in the form

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-jk\frac{2\pi}{N}n} \tag{3.43}$$

for the unitless frequency index $k = 0, 1, \cdots, N-1$ which can be related according to the previous notes to

- digital frequency in [rad] : $\omega_k = k2\pi/N \in \langle 0, 2\pi)$
- digital frequency in [Hz] : $f_k = \omega_k/(2\pi) = k/N \in \langle 0, 1)$
- analogue frequency in [rad/s] : $\Omega_k = \omega_k/T_s \in \langle 0, 2\pi/T_s)$

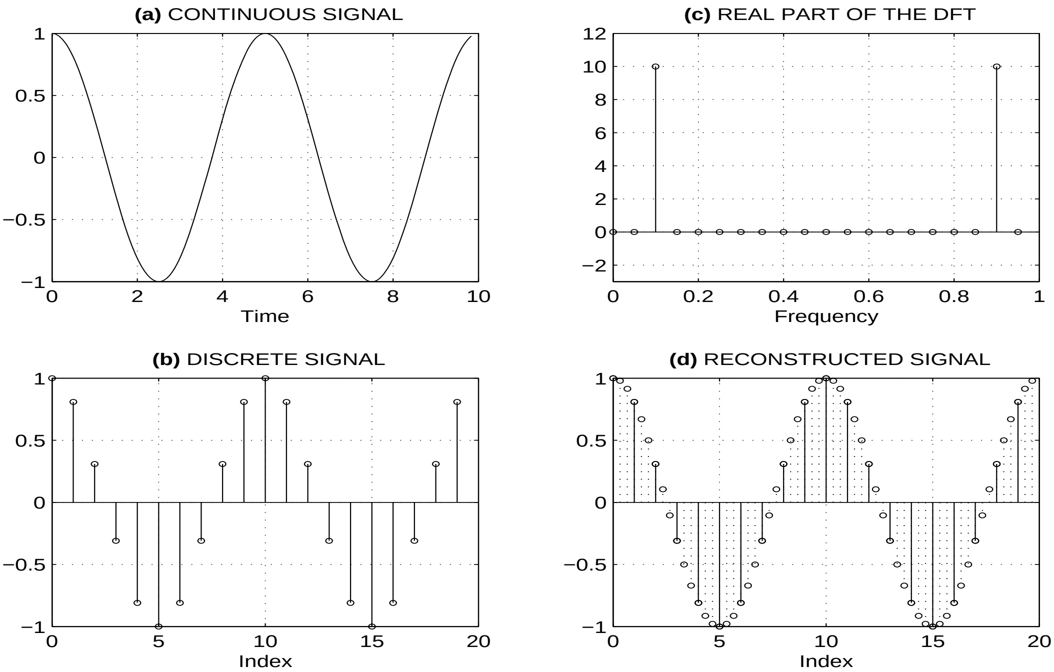

FIGURE 3.11. Signal decomposition and reconstruction: (**a**) Continuous signal $x_a(t) = cos(\Omega_a t)$ for $\Omega_a = 2\ \pi\ f_a \pi$ [rad/s] for $f_a = 0.2[Hz]$ and $t \in < 0, 10)$ [s] (**b**) Discrete signal $x(n) = x_a(n\ T_s)$, $n = 0, 1, \cdots, N-1$ for sampling period $T_s = 0.5$ [s] ($f_s = 1/T_s = 2[Hz]$, $N = 20$ and resulting normalized digital frequency $f_d = f_a/f_s = 0.1$ (**c**) Real part of $X(k)$ defined as a DFT of $\{x(n)\}$ and presented for $k = 0, 1, \cdots, N-1$ (**d**) Result of the inverse DFT of $X(k)$ for signal reconstruction combined with digital interpolation

Using further for simplicity real even sequence $\{x(n)\}_{n=0}^{N-1}$ with the number of its values equal to the multiple of the signal period the evaluation of the DFT results in the real even sequence $\{X(k)\}_{k=0}^{N-1}$ [30, p.93]. The whole process of sampling and analysis for such a harmonic sequence with its digital frequency $\omega_d = 0.2\pi$ ($f_d = 0.1$) and $N = 20$ samples is given in Fig. 3.11 (a), (b) and (c). The result of the DFT is presented for $\omega_k = k2\pi/N, k = 0, 1, \cdots, N/2$ only taking into account that evaluations for indices greater than $k = N/2$ related to frequencies greater then $\omega_k = \pi$ are redundant owing to the periodicity properties of the DFT.

SIGNAL RECONSTRUCTION is based upon the inverse discrete Fourier transform in the form

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-jk\frac{2\pi}{N}n} \tag{3.44}$$

for $n = 0, 1, \cdots, N-1$. Using the previous example it is possible to apply this equation to the sequence in Fig. 3.11 (d) given by solid lines. To obtain more values of the reconstructed sequence it is possible to use *digital interpolation* for evaluation of values between these samples given in Fig. 3.11 (d) by dotted lines. The principle of this interpolation [40, p.80] is based on the following statements with their graphic interpretation restricted in Fig. 3.12 for an even sequence with real part of the DFT only

- Real sequence $\{x(n)\}_{n=0}^{N-1}$ derived from a band limited continuous signal $x_a(t)$ with sampling T_s has its DFT $X(k)$ decreasing to zero for $k \to N/2$ and owing to properties of the discrete Fourier transform $X(N-k) = conj(X(k))$ for $k = 0, 1, \cdots, N-1$.

- Real sequence $\{v(m)\}_{m=0}^{M-1}$ where $M = N \cdot N_s$ derived from the same band limited continuous signal $x_a(t)$ with sampling T_s/N_s has its DFT $V(k)$ closely related to the original one with $N \cdot (N_s - 1)$ inserted zero values as no new frequency components are present

$$\{V(k)\}_{k=0}^{M-1} = \frac{M}{N}\{X(0), X(1), \cdots, X(N/2-1), 0, 0, \cdots, 0, X(N/2), \cdots, X(N-1)\} \tag{3.45}$$

Constant M/N introduced in Eq. (3.45) is caused by the different length of sequences $\{x(n)\}$ and $\{v(n)\}$ which affects the multiplication factor in the definition of the inverse DFT. Fig. 3.12 (b) and (e) explain that the analogue resolution frequencies are the same for the DFT of both sequences $\{x(n)\}$ and $\{v(n)\}$. Computer processing of the digital interpolation (for even N) can be based upon the Algorithm 3.6 with all indices shifted by one to have their positive values only. Similar process can be designed for odd N.

We have supposed till now the digital frequency ω_d slow enough enabling signal decomposition and reconstruction as well. It is obvious from Fig. 3.11 that when frequency ω_d is

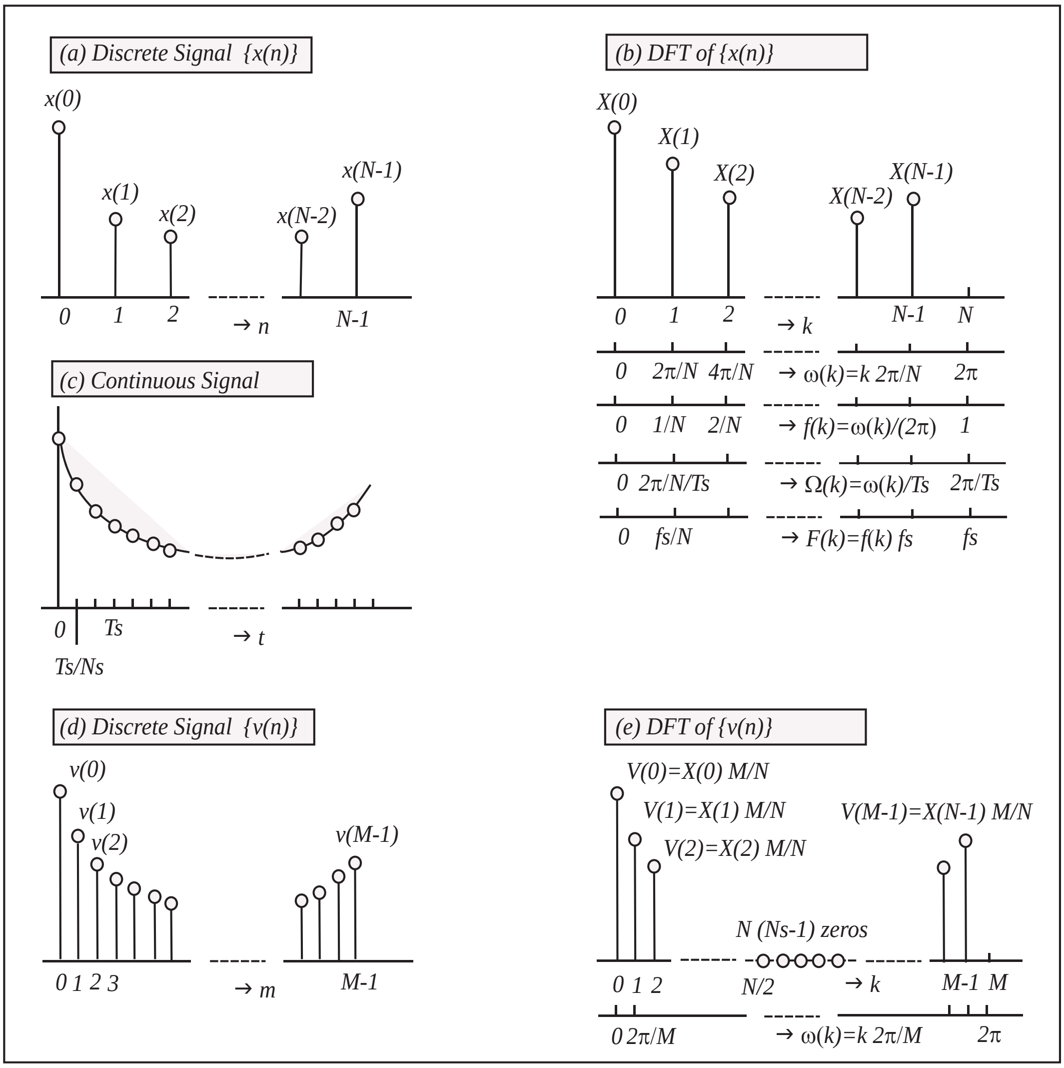

FIGURE 3.12. Principle of the digital interpolation of signal $\{x(n)\}_{n=0}^{N-1}$ =IDFT$[\{X(k)\}_{k=0}^{N-1}]$ by the inverse discrete Fourier transform of $\{V(k)\}_{k=0}^{M-1}$ for $M = N \cdot N_s$ with N_s standing for the number of subsampling intervals.

Algorithm 3.6 *Digital interpolation for signal reconstruction using the inverse DFT*

- definition of vector $\mathbf{x} = [x(1), x(2), \cdots, x(N)]$ and the subsampling index NS
- discrete Fourier transformm evaluation
  $\mathbf{X} = \text{fft}\ (\mathbf{x})$
  defining sequence $\mathbf{X} = [X(1), X(2), \cdots, X(N/2), X(N/2+1) \cdots, X(N)]$
- new sequence definition of the length $M = N \cdot N_s$ with inserted zero values
  $\mathbf{V} = \frac{M}{N}\ [\mathbf{X}(1 : N/2), \text{zeros}\ (1, N * (NS-1), \mathbf{X}(N/2+1 : N)]$
- inverse discrete Fourier transform
  $\mathbf{y} = \text{ifft}\ (\mathbf{V})$

changing from zero to π the DFT is able to distinguish this frequency component (evaluating its complex conjugate in the range $\langle \pi, 2\pi)$ as well). But when ω_d is grater than π the interpretation is not unique already. This situation is given in Fig. 3.13 for $\omega_d = 1.8\pi$ [rad]. Values of this discrete signal are the same as those in Fig. 3.11 for $\omega_d = 0.2\pi$ [rad] and the reconstruction process provides signal given in Fig. 3.13 with its digital frequency in range $\langle 0, \pi)$ different from the original signal in this case. This frequency reduction is often presented as *aliasing* and it results in the signal reconstruction of the lowest possible frequency component defined by the given sequence as given in Fig. 3.14

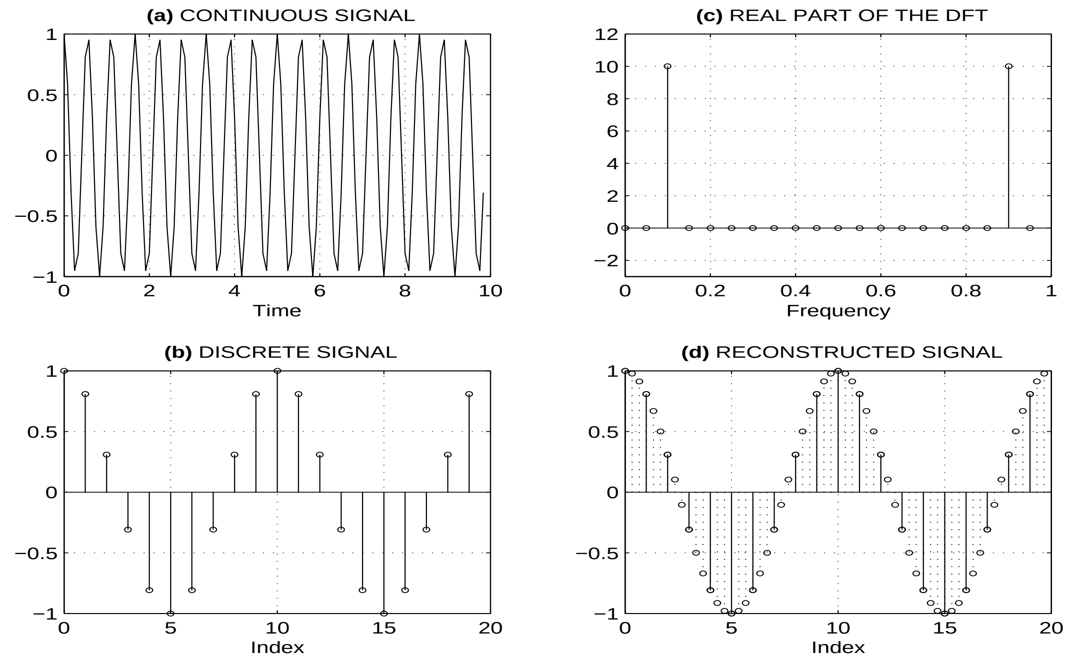

FIGURE 3.13. Signal decomposition and reconstruction: (**a**) Continuous signal $x_a(t) = cos(\Omega_a t)$ for $\Omega_a = 2\ \pi\ f_a \pi$ [rad/s] for $f_a = 1.8[Hz]$ and $t \in < 0, 10)$ [s] (**b**) Discrete signal $x(n) = x_a(n\ T_s)$, $n = 0, 1, \cdots, N-1$ for sampling period $T_s = 0.5$ [s] ($f_s = 1/T_s = 2[Hz]$, $N = 20$ and resulting normalized digital frequency $f_d = f_a/f_s = 0.9$ (**c**) Real part of $X(k)$ defined as a DFT of $\{x(n)\}$ and presented for $k = 0, 1, \cdots, N-1$ (**d**) Result of the inverse DFT of $X(k)$ for signal reconstruction combined with digital interpolation

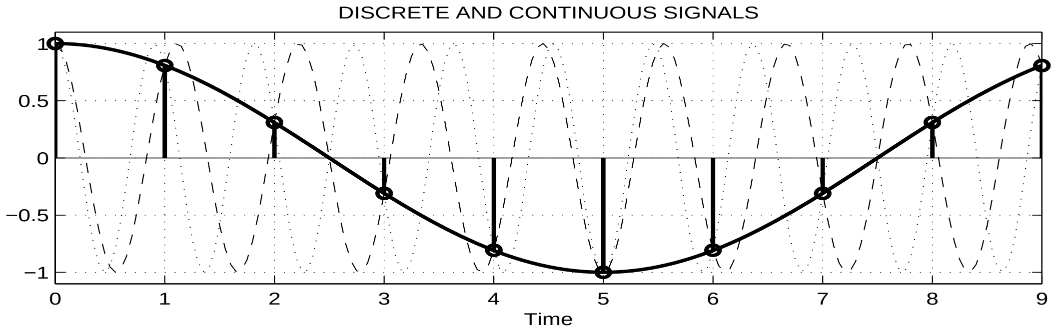

FIGURE 3.14. Continuous signal $x_a(t) = cos(2\pi\, f_a\, t)$ for $f_a = 0.1, 0.9$ and 1.1 [Hz] resulting in the same discrete signal $x(n) = cos(2\pi\, f_d\, n)$ for sampling period $T_s = 1$ and $f_d = f_a$ causing aliasing.

In general any continuous signal of frequency $\Omega_a > \pi/T_s$ is aliased to the frequency range $\langle 0, \pi/T_s)$. To avoid such an aliasing it is necessary to choose the sampling period $T_s \langle \pi/\Omega_a$ or the sampling frequency $f_s > 2f_a$ confirming the sampling theorem presented in the previous chapter. The highest frequency component of a signal implies in this case the necessary sampling rate for further digital signal processing. To reduce the number of the discrete signal values it is sometimes possible to reduce the high frequency components in the analogue signal already and to sample such a prefiltered signal.

3.4 Method of the Least Squares and the Gradient Method

The previous mathematical background was devoted to various methods of signal and system description based on discrete transforms. Further mathematical methods enabling signal and system modelling are based upon the parameters estimation by the least square method. This principle is essential in many engineering applications including signal approximation, prediction and adaptive filtering as well. Its specific modifications will be discussed in further chapters and we shall summarize here basic principles only resulting in finite and iterative methods [43], [40], [12] using nonorthogonal and orthogonal basis functions during the search process and parameters evaluation [25], [20].

3.4.1 General Principle of the Least Square Method

Basic principle of the least square method can be explained on approximation of given values $\{x(n), y(n)\}_{n=0}^{N-1}$ by a linear combination of basis functions $\{g_j(x)\}_{j=0}^{M-1}$ in the form

$$f(x) = \sum_{j=0}^{M-1} w_j g_j(x) \tag{3.46}$$

Main problems of the approximation can be summarized in the following items

- estimation of the general form of function (3.46)
- evaluation of coefficients $w_0, w_1, \cdots, w_{M-1}$ by a chosen method

The first problem can be often solved taking into account physical principle of approximated values and the second one presumes the choice of a proper criterium.

Function $f(x)$ is often looked upon as a continuous function of a variable x but in digital signal processing applications its discrete values are used only defined on a sequence

$\{x(n)\}_{n=0}^{N-1}$. This special case of the *approximation problem* is often referred to as *signal modelling*. In case that we further assume $g_j(x(n)) = x(n-j)$ it is possible to rewrite expression 3.46 to the form

$$f(n) = \sum_{j=0}^{M-1} w_j x(n-j) \tag{3.47}$$

with $f(n)$ standing for $f(x(n))$ in fact. This specific case corresponds to the system representation by the impulse response mentioned before which implies that classical methods of the least square approximation can be applied in both cases. Comparison of a general and specific function defined by Eq. (3.46) and (3.47) for a given set of values $\{x(n)\}$ is presented in Fig. 3.15.

The *method of the least squares* is used for the minimization of the total squared error between given and approximated values visualized in Fig. 3.16 for a chosen example and having generally form

$$J(w_0, w_1, \cdots, w_{M-1}) = \sum_{n=0}^{N-1} \epsilon(n)^2 = \sum_{n=0}^{N-1} (y(n) - f(x(n))^2 = \sum_{n=0}^{N-1} (y(n) - \sum_{j=0}^{M-1} w_j g_j(x(n)))^2 \tag{3.48}$$

As J is a function of variables $w_0, w_1, \cdots, w_{M-1}$ it is possible to find their values minimizing the total sum (3.48) standing for the error-performance surface in Fig. 3.16 and defining coefficients of function (3.46) in this way.

Theorem 3.1 *Let us assume a sequence $\{x(n), y(n)\}_{n=0}^{N-1}$. Then the coefficients $\{w_j\}_{j=0}^{M-1}$ of the approximation function (3.46) for a given basis $\{g_j(x)\}_{j=0}^{M-1}$ are given by the solution of the set of M linear algebraic equations*

$$\mathbf{R}\mathbf{w} = \mathbf{p} \tag{3.49}$$

where

$$\mathbf{R} = \begin{bmatrix} \sum g_0(x(n))g_0(x(n)) & \cdots & \sum g_0(x(n))g_{M-1}(x(n)) \\ & \cdots\cdots\cdots & \\ \sum g_{M-1}(x(n))g_0(x(n)) & \cdots & \sum g_{M-1}(x(n))g_{M-1}(x(n)) \end{bmatrix},$$

$$\mathbf{w} = \begin{bmatrix} w_0 \\ \cdots \\ w_{M-1} \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} \sum y(n)g_0(x(n)) \\ \cdots \\ \sum y(n)g_{M-1}(x(n)) \end{bmatrix}$$

with all summations for $n = 0, 1, \cdots, N-1$.

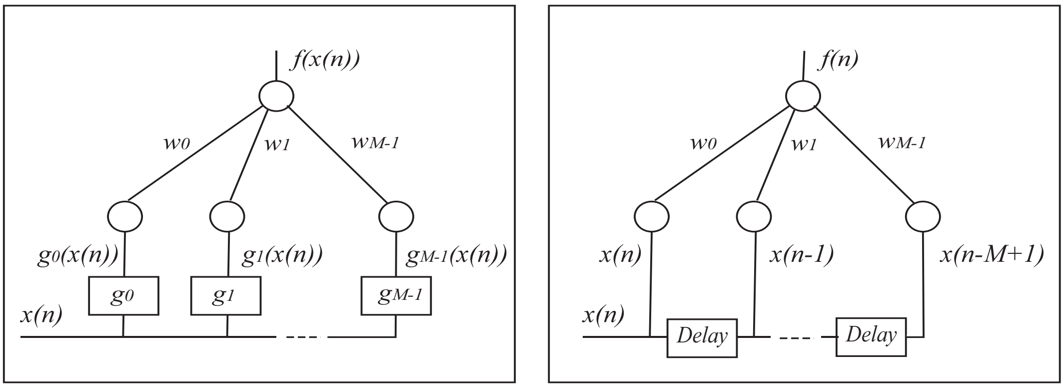

FIGURE 3.15. Comparison between approximation function from the general and signal processing point of view

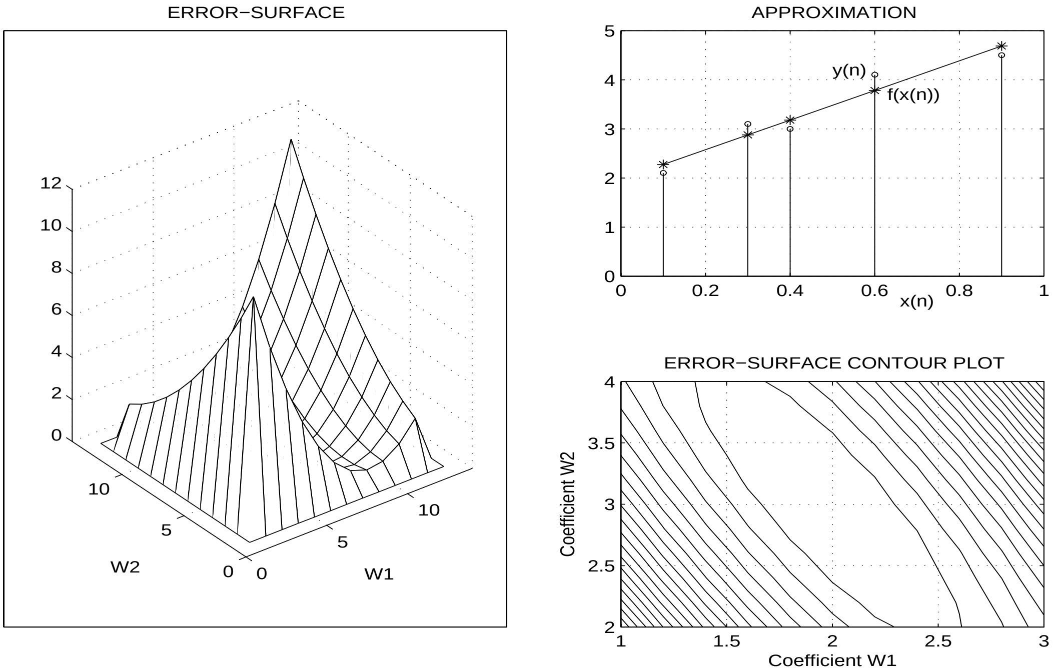

FIGURE 3.16. Error-performance surface for the linear approximation of a given sequence of $N = 5$ values by function $f(x) = w_0 + w_1 x$ by the least squares method and plot of given and evaluated values

Proof: To minimize the sum of squares in form (3.48) it is necessary to evaluate its partial derivatives with respect to coefficients $\{w_j\}_{j=0}^{M-1}$ and to put them equal to zero which means that

$$\frac{\partial J}{\partial w_i} \equiv 2\sum_{n=0}^{N-1}(y(n) - \sum_{j=0}^{M-1} w_j g_j(x(n)) g_i(x(n))) = 0$$

for $i = 0, 1, \cdots, M-1$. Rearranging this equation we shall obtain

$$\sum_{j=0}^{M-1} w_j \sum_{n=0}^{N-1} g_j(x(n)) g_i(x(n)) = \sum_{n=0}^{N-1} y(n) g_i(x(n))$$

The last expression is equivalent to (3.49) and it represents the set of M linear algebraic equations defining coefficients $w_0, w_1, \cdots, w_{M-1}$.

Example 3.9 *Evaluate coefficients of the approximation function in the form*

$$f(x) = w_0 + w_1 x$$

for a given sequence $\{x(n), y(n)\}_{n=0}^{N-1}$.

Solution: Using Eq. (3.48) it is possible to express the sum of squares in the form

$$J(w_0, w_1) = \sum_{n=0}^{N-1}(y(n) - w_0 - w_1 x(n))^2$$

The condition for minimizing this expression can be stated in the form

$$\frac{\partial J}{\partial w_0} \equiv -2\sum_{n=0}^{N-1}(y(n) - w_0 - w_1 x(n)) = 0$$

$$\frac{\partial J}{\partial w_1} \equiv -2\sum_{n=0}^{N-1}(y(n) - w_0 - w_1 x(n)) x(n) = 0$$

which results in the set of equation

$$\begin{aligned} w_0 N + w_1 \sum x(n) &= \sum y(n) \\ w_0 \sum x(n) + w_1 \sum x(n)^2 &= \sum x(n) y(n) \end{aligned}$$

with all summations for $n = 0, 1, \cdots, N-1$ defining coefficients w_0 and w_1. Graphic results of this example for a given sequence of values is presented in Fig. 3.16.

The set (3.49) of linear algebraic equations is not well conditioned which might cause numerical problems in its solution. It is one of reasons why orthogonal basis functions are used as well.

Definition 3.3 *The sequence of functions $\{p_j(x)\}_{j=0}^{M-1}$ is said to be orthogonal with respect to a given sequence $\{x(n)\}_{n=0}^{N-1}$ in case that*

$$\sum_{n=0}^{N-1} p_i(x(n)) p_j(x(n)) \begin{cases} = 0 & \text{for } i \neq j \\ \neq 0 & \text{for } i = j \end{cases} \tag{3.50}$$

The sum (3.50) represents scalar multiplication in fact referred to as $(p_i(x), p_j(x))$ very often which substantially simplifies the approximation problem stated in the next theorem.

Theorem 3.2 *Let as assume a sequence $\{x(n), y(n)\}_{n=0}^{N-1}$. Then the coefficients $\{w_j\}_{j=0}^{M-1}$ of the approximation function*

$$f(x) = \sum_{j=0}^{M-1} w_j p_j(x) \tag{3.51}$$

for given orthogonal basis functions $\{p_j(x\}_{j=0}^{M-1}$ with respect to the sequence $\{x(n)\}_{n=0}^{N-1}$ are defined by relation

$$w_j = \frac{\sum_{n=0}^{N-1} y(n) p_j(x(n))}{\sum_{n=0}^{N-1} p_j(x(n)) p_j(x(n))} \tag{3.52}$$

for $j = 0, 1, \cdots, M-1$.

Proof of this statement is based upon that of Theorem 3.1 with the matrix G reduced to the diagonal matrix with zero nondiagonal elements owing to the definition of orthogonal functions. As no set of equations is solved in this case it is possible very simply to evaluate any further coefficient w_j to improve the approximation with no change of coefficients defined before.

A typical example of the error-performance surface in this case is presented on Fig 3.17 for the linear approximation. The comparison of this sketch with that in Fig. 3.16 for the nonorthogonal bases functions illustrates that orthogonality changes positions of axis only with no affect to a very flat minimum of the surface.

Definition of the set of orthogonal basis functions $\{p_j(x)\}_{j=1}^{M-1}$ can be based upon the *Gramm - Schmidt* process [25] applied to the nonorthogonal set of functions $\{g_j(x)\}_{j=0}^{M-1}$. The whole process can be summarized in the following way using the notation for scalar multiplication mentioned before

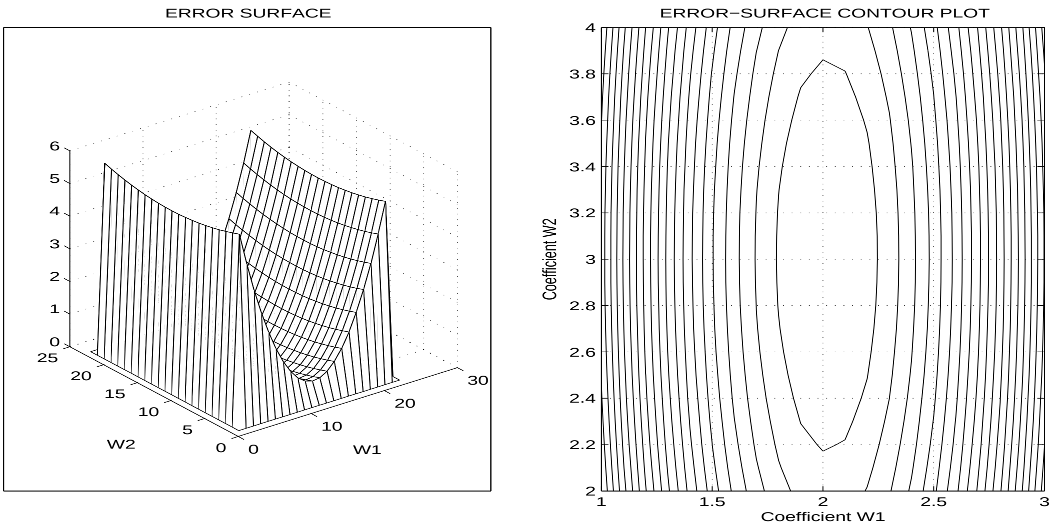

FIGURE 3.17. Error-performance surface for linear approximation of a given sequence of $N = 5$ values $\{x(n), y(n)\}$ by function $f(x(n)) = w_0 + w_1(x(n) - mean(\mathbf{x}))$ involving the set of orthogonal basis functions $\{1, \mathbf{x} - mean(\mathbf{x})\}$ for given values $\{x(n)\}$

- definition of
$$p_0(x) = g_0(x)$$
- estimation of
$$p_1(x) = g_1(x) - \Lambda_{01} p_0(x)$$
orthogonal to $p_1(x)$ implying that
$$(p_1(x), p_0(x)) \equiv (g_1(x), p_0(x)) - \Lambda_{01}(p_0(x), p_0(x)) = 0$$
$$\Lambda_{01} = \frac{(g_1(x), p_0(x))}{(p_0(x), p_0(x))}$$
- estimation of
$$p_2(x) = g_2(x) - \Lambda_{02} p_0(x) - \Lambda_{12} p_1(x)$$
orthogonal to $p_1(x)$ and $p_2(x)$ implying that
$$(p_2(x), p_0(x)) \equiv (g_2(x), p_0(x)) - \Lambda_{02}(p_0(x), p_0(x)) - \Lambda_{12}(p_1(x), p_0(x)) = 0$$
$$\Lambda_{02} = \frac{(}{g_2(x), p_0(x))}(p_0(x), p_0(x))$$
and
$$(p_2(x), p_1(x)) \equiv (g_2(x), p_1(x)) - \Lambda_{02}(p_0(x), p_1(x)) - \Lambda_{12}(p_1(x), p_1(x)) = 0$$
$$\Lambda_{12} = \frac{(}{g_2(x), p_1(x)})(p_1(x), p_1(x))$$

The same process can be applied for further functions in the same way as well.

Example 3.10 *Evaluate coefficients of the approximation function in the form*

$$f(x) = w_0 p_0(x) + w_1 p_1(x)$$

for a given sequence $\{x(n), y(n)\}_{n=0}^{N-1}$ *and orthogonal bases functions* $\{p_0(x), p_1(x)\}$ *defined by nonorthogonal functions* $g_0(x) = 1$ *and* $g_1(x) = x$.

Solution: Using the Gramm-Schmidt process described before it is possible to define

$$p_0(x) = g_0(x) = 1$$
$$p_1(x) = g_1(x) - \Lambda_{01} p_0(x)$$

where

$$\Lambda_{01} = \frac{(g_1(x), p_0(x))}{(p_0(x), p_0(x))} = \frac{\sum_{n=0}^{N-1} x(n)}{N}$$

The approximation function has therefore the following form

$$f(x) = w_0 + w_1(x - \Lambda_{01})$$

where according to Eq. (3.52)

$$\begin{aligned} w_0 &= \frac{1}{N}\sum_{n=0}^{N-1} y(n) \\ w_1 &= \frac{\sum_{n=0}^{N-1} y(n)(x(n) - \Lambda_{01})}{\sum_{n=0}^{N-1}(x(n) - \Lambda_{01})^2} \end{aligned}$$

Graphic results of this example for a given sequence of values is presented in Fig. 3.16.

Examples 3.9 and 3.10 explain that the same approximation function can be evaluated in two possible ways. Nonorthogonal basis functions results in the set of algebraic equations while orthogonal basis functions enable direct evaluation of coefficients after the previous orthogonalization process.

3.4.2 The Steepest Descent Method

In case of the linear approximation the basic method described in the previous section can be used. In more general case and nonlinear approximation function other methods must be applied. We shall describe now very briefly the *gradient method* used very often in many applications involving optimization problems of various objective functions.

The total squared error given by Eq. (3.48) presented before is a function of M variables $\{w_0, w_1, \cdots, w_{M-1}\}$ in the form

$$J(w_0, w_1, \cdots, w_{M-1}) = \sum_{n=0}^{N-1}(y(n) - \sum_{j=0}^{M-1} w_j g_j(x(n)))^2 \tag{3.53}$$

or in the equivalent matrix notation

$$J(\mathbf{w}) = (\mathbf{y} - \mathbf{G}'\mathbf{w})'(\mathbf{y} - \mathbf{G}'\mathbf{w}) \tag{3.54}$$

where

$$\begin{aligned} \mathbf{w} &= [w_0, w_1, \cdots, w_{M-1}]' \\ \mathbf{y} &= [y(0), y(1), \cdots, y(N-1)]' \\ \mathbf{x} &= [x(0), x(1), \cdots, x(N-1)]' \end{aligned}$$

and

$$\mathbf{G} = \begin{bmatrix} g_0(\mathbf{x}') \\ \cdots \\ g_{M-1}(\mathbf{x}') \end{bmatrix} = \begin{bmatrix} g_0(x(0)) & \cdots & g_0(x(N-1)) \\ & \cdots\cdots\cdots & \\ g_{M-1}(x(0)) & \cdots & g_{M-1}(x(N-1)) \end{bmatrix}$$

with an apostrophe standing for matrix or vector transposition.

To find the optimum vector $\mathbf{w}$ defining the minimum value of function (3.54) assumes the evaluation of the *gradient vector*

$$\frac{\partial J(w_0, \cdots, w_{M-1})}{\partial w_i} = 2\sum_{n=0}^{N-1}(y(n) - \sum_{j=0}^{M-1} w_j g_j(x(n)))g_i(x(n)) \tag{3.55}$$

resulting in the following matrix notation

$$\frac{\partial J(\mathbf{w})}{\partial w_i} = -2\mathbf{G}(i,:)(\mathbf{y} - \mathbf{G}'\mathbf{w}) \tag{3.56}$$

where $\mathbf{G}(i,:)$ stands for the i-th row of matrix $\mathbf{G}$ and $i = 0, 1, \cdots, M-1$ or in the compact form

$$\mathbf{q} = \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} = -2\mathbf{G}(\mathbf{y} - \mathbf{G}'\mathbf{w}) = \mathbf{2}(\mathbf{R}\mathbf{w} - \mathbf{p}) \tag{3.57}$$

where

$$\mathbf{R} = \mathbf{G}\mathbf{G}', \qquad \mathbf{p} = \mathbf{G}\mathbf{y} \tag{3.58}$$

The optimum vector $\mathbf{w} = \mathbf{w}^*$ has such values for which its gradient is equal to zero resulting in Eq. (3.49) in the form

$$\mathbf{R}\mathbf{w}^* = \mathbf{p} \tag{3.59}$$

discussed in the previous section already.

Using this notation for the optimum gradient vector it is possible to use it in Eq. (3.54) which provides another expression for the sum of squares based on the weight deviation vector

$$\mathbf{v} = \mathbf{w} - \mathbf{w}^* \tag{3.60}$$

As

$$\begin{aligned} J(\mathbf{w}) &= (\mathbf{y} - \mathbf{G}'\mathbf{w})'(\mathbf{y} - \mathbf{G}'\mathbf{w}) = (\mathbf{y}' - \mathbf{w}'\mathbf{G})(\mathbf{y} - \mathbf{G}'\mathbf{w}) = \\ &= \mathbf{y}'\mathbf{y} - \mathbf{w}'\mathbf{G}\mathbf{y} - \mathbf{y}'\mathbf{G}'\mathbf{w} + \mathbf{w}'\mathbf{G}\mathbf{G}'\mathbf{w} \end{aligned}$$

it is possible to use (3.58) and (3.60) to find

$$\begin{aligned} J(\mathbf{w}) &= \mathbf{y}'\mathbf{y} - (\mathbf{v} + \mathbf{w}^*)'\mathbf{p} - \mathbf{p}'(\mathbf{v} + \mathbf{w}^*) + (\mathbf{v} + \mathbf{w}^*)'\mathbf{R}(\mathbf{v} + \mathbf{w}^*) = \\ &= \mathbf{y}'\mathbf{y} - \mathbf{v}'\mathbf{p} - (\mathbf{w}^*)'\mathbf{p} - \mathbf{p}'\mathbf{v} - \mathbf{p}'\mathbf{w}^* + \mathbf{v}'\mathbf{R}\mathbf{v} + \mathbf{v}'\mathbf{R}\mathbf{w}^* + (\mathbf{w}^*)'\mathbf{R}\mathbf{v} + (\mathbf{w}^*)'\mathbf{R}\mathbf{w}^* \end{aligned}$$

Using (3.59) it follows that $\mathbf{v}'\mathbf{R}\mathbf{w}^* = \mathbf{v}'\mathbf{p}$ and $(\mathbf{w}^*)'\mathbf{R}\mathbf{w}^* = (\mathbf{w}^*)'\mathbf{p}$ and as $\mathbf{R}' = \mathbf{R}$ it is possible to express $(\mathbf{w}^*)'\mathbf{R}\mathbf{v} = \mathbf{p}'\mathbf{v}$ which results in the following relation

$$J(\mathbf{w}) = \mathbf{y}'\mathbf{y} - (\mathbf{w}^*)'\mathbf{R}\mathbf{w}^* + \mathbf{v}'\mathbf{R}\mathbf{v} \tag{3.61}$$

or

$$J(\mathbf{w}) = J_{min}(\mathbf{w}) + \mathbf{v}'\mathbf{R}\mathbf{v} \tag{3.62}$$

The last expression is used very often to evaluate the error-performance surface around its minimum value and to find gradients for further processing as well.

We can visualize the dependence of the squared error on elements of vector $\mathbf{w}$ by a sketch in Fig. 3.18 for $M = 2$ elements only and to design an alternative procedure to the finite least square method referred to as the method of the *steepest descent* summarised in Algorithm 3.7 for the approximation problem.

Example 3.11 *Evaluate the coefficients of the approximation function in the form*

$$f(x) = w_0 + w_1 x \tag{3.65}$$

for a given sequence $\{x(n), y(n)\}_{n=0}^{N-1}$.

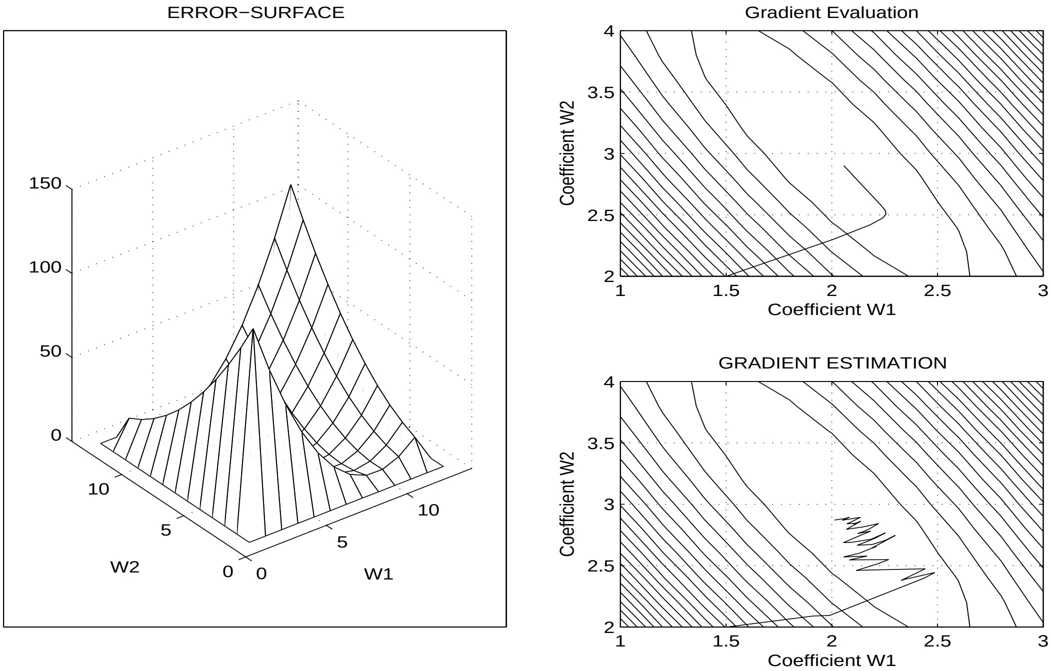

FIGURE 3.18. Error-performance surface for the linear approximation of a given sequence $\mathbf{y} = 2 + 3 * \mathbf{x}$ with the additive random noise and random values of vector $\mathbf{x} = [x(0), x(1), \cdots, x(N-1)]'$ for $N = 50$ by function $f(x) = w_0 + w_1 x$ and results of the steepest descent search with gradient evaluated both for the whole set of given values and estimated separately for each of approximated values with the initial estimate $\mathbf{w} = [1.5, 2]$

.

Algorithm 3.7 *The steepest descent method applied for approximation of N values $\mathbf{y} = y(\mathbf{x})$ for $\mathbf{x} = [x(0), \cdots, x(N-1)]'$ by sequence $\mathbf{f} = \mathbf{w}'\mathbf{G}$ with weights $\mathbf{w} = [w_0, \cdots, w_{M-1}]'$ minimizing the objective function*

$$J(\mathbf{w}) = (\mathbf{y} - \mathbf{G}'\mathbf{w})'(\mathbf{y} - \mathbf{G}'\mathbf{w})$$

for a given set $\{g_0, \cdots, g_{M-1}\}$ of M basis functions defining matrix

$$\mathbf{G} = [g_0(\mathbf{x}'), \cdots, g_{M-1}(\mathbf{x}')]'$$

- definition of vectors $\mathbf{x}, \mathbf{y}$ and matrix $\mathbf{G}$ of values of basis functions
- estimation of the initial guess of coefficients $\mathbf{w}$
- iterative evaluation of
  - the gradient vector
  $$\mathbf{q} = -2 * \mathbf{G} * (\mathbf{y} - \mathbf{G}' * \mathbf{w}) \tag{3.63}$$
  - new estimate of coefficients in direction opposite to that of the gradient vector for a given convergence factor c
  $$\mathbf{w} = \mathbf{w} - c * \mathbf{q} \tag{3.64}$$

Solution: Matrix **G** defined by Eq. (3.54) has values

$$\mathbf{G} = \begin{bmatrix} \mathbf{x}' \cdot \wedge 0 \\ \mathbf{x}' \cdot \wedge 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x(0) & x(1) & \cdots & x(N-1) \end{bmatrix} \tag{3.66}$$

with symbol $\cdot\wedge$ defining that each element of a given vector is raised to the given power and it implies values of gradient vector (3.57) used in the iterative Algorithm **??**. Results for a chosen artificial sequence $\mathbf{y} = 2 + 3 * \mathbf{x}$ with the additive random noise and random values of vector $\mathbf{x} = [x(0), x(1), \cdots, x(N-1)]'$ for $N = 50$ are presented in Fig. 3.18 for gradient evaluated for the whole set of given values and initial estimate of weights $\mathbf{w} = [1.5, 2]$.

The same principle of the gradient search can be applied in various modifications to any objective function and other problems as well. In context of signal processing and adaptive filtering the method of the steepest descent is modified to much simple form but a slower process of convergence, too [43, 12].

Let us consider a single squared error of Eq. (3.53) in the form

$$\varepsilon(n)^2 = (y(n) - \sum_{j=0}^{M-1} w_j g_j(x(n)))^2 \tag{3.67}$$

as an estimate of the mean of squared error used for the gradient evaluation before [43, p.100]. The gradient estimate for each n can be then written in the form

$$\hat{\mathbf{q}}(n) = \begin{bmatrix} \frac{\partial \varepsilon(n)^2}{\partial w_0} \\ \cdots \\ \frac{\partial \varepsilon(n)^2}{\partial w_{M-1}} \end{bmatrix} = 2\varepsilon(n) \begin{bmatrix} \frac{\partial \varepsilon(n)}{\partial w_0} \\ \cdots \\ \frac{\partial \varepsilon(n)}{\partial w_{M-1}} \end{bmatrix} = -2\varepsilon(n) \begin{bmatrix} g_0(x(n)) \\ \cdots \\ g_{M-1}(x(n)) \end{bmatrix} \tag{3.68}$$

It is obvious that

$$\mathbf{q} = \sum_{n=0}^{N-1} \hat{\mathbf{q}}(n) \tag{3.69}$$

enabling to evaluate the gradient from its estimates. The whole process for such a modified gradient method is given in Algorithm 3.8.

The *convergence factor c* has the same meaning as before and it regulates the speed and stability of convergence. As the estimates of the gradient vector are imperfect it is possible to expect noisy adaptive process not following the true line of the steepest descent. Results for the previous example with $M = 2$ elements of vector **w** only are given in Fig. 3.18. The choice of orthogonal basis functions can improve the whole process of adaptation.

Similar method of that described before can be applied in case of signal processing applications. Defining the basis functions in $g_j(x(n)) = x(n-j)$ it is possible to find the estimate of values $\{y(n)\}$ in the same way for the objective function in the form

$$J(w_0, w_1, \cdots, w_{M-1}) = \sum_{n=0}^{N-1} \varepsilon(n)^2 = \sum_{n=0}^{N-1} (y(n) - \sum_{j=0}^{N-1} w_j x(n-j))^2 \tag{3.72}$$

As the sequence of values $\{x(n), y(n)\}$ is usually very long the approximate gradient method is used very often. The estimate of the gradient vector given by Eq. (3.72) can be then stated in the following form

$$\hat{\mathbf{q}}(n) = -2\varepsilon(n) \begin{bmatrix} x(n) \\ x(n-1) \\ \cdots \\ x(n-M+1) \end{bmatrix} \tag{3.73}$$

Algorithm 3.8 *The gradient search applied for approximation of* N *values* $\mathbf{y} = y(\mathbf{x})$ *for* $\mathbf{x} = [x(0), \cdots, x(N-1)]'$ *by sequence* $\mathbf{f} = \mathbf{w}'\mathbf{G}$ *with weights* $\mathbf{w} = [w_0, \cdots, w_{M-1}]'$ *minimizing the objective function*

$$J(\mathbf{w}) = (\mathbf{y} - \mathbf{G}'\mathbf{w})'(\mathbf{y} - \mathbf{G}'\mathbf{w})$$

for a given set $\{g_0, \cdots, g_{M-1}\}$ *of* M *basis functions defining matrix*

$$\mathbf{G} = [g_0(\mathbf{x}'), \cdots, g_{M-1}(\mathbf{x}')]'$$

- definition of vectors $\mathbf{x}$, $\mathbf{y}$ and matrix $\mathbf{G}$ of values of basis functions
- estimation of the initial guess of coefficients $\mathbf{w}$
- iterative evaluation for each n of
  - the estimate of the gradient vector

$$\hat{\mathbf{q}}(n) = -2\mathbf{G}(:, n) * (y(n) - \mathbf{G}(:, n)' * \mathbf{w}) \tag{3.70}$$

  - new estimate of coefficients in direction opposite to that of the gradient vector for a given convergence factor c

$$\mathbf{w} = \mathbf{w} - c * \hat{\mathbf{q}} \tag{3.71}$$

defining the iterative process

$$\mathbf{w}_{new} = \mathbf{w}_{old} - c\hat{\mathbf{q}}(n) \tag{3.74}$$

The sample process of adaptation for $M = 2$ weights is presented in Fig. 3.19.

The mean least square principle and gradient methods are essential in many signal processing applications and they are closely related to the classical Newton method as well. Their more detail discussion will be presented further.

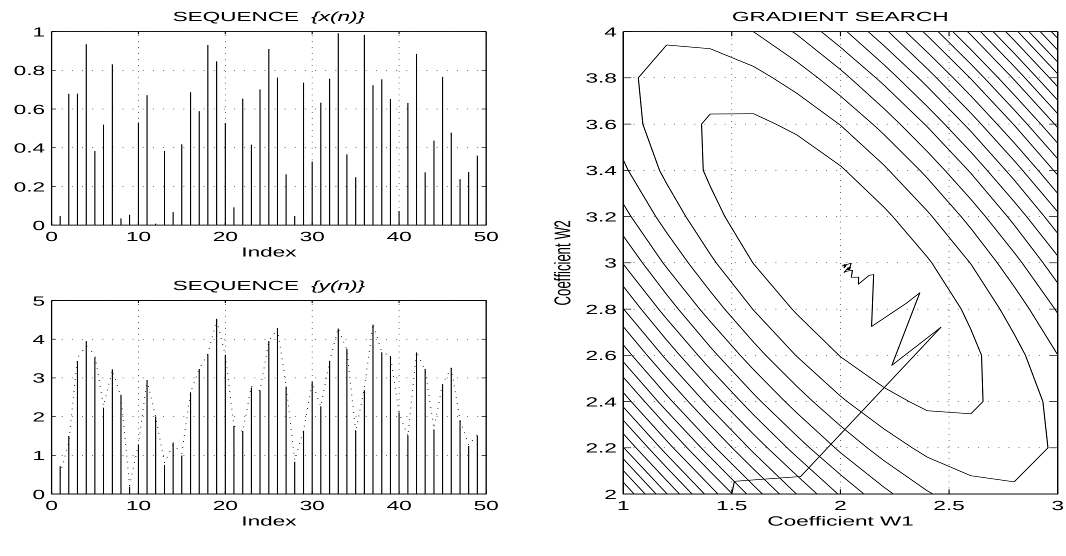

FIGURE 3.19. Signal modelling of a chosen sequence $\mathbf{y} : y(n) = 3x(n) + 2x(n-1)$ with the additive random noise for $N = 50$ random values of vector $\mathbf{x}$ by values $\{\mathbf{f} : f(n) = w_0 x(n) + w_1 x(n-1)\}$ with weights continuously updated using the gradient estimate and initial guess of vector $\mathbf{w} = [1.5, 2]$

3.5 Summary

Z-transform stands for a basic mathematical tool in signal processing methods enabling representation of a signal $\{x(n)\}$ in complex domain by a function $X(z)$ of complex variable z. *Direct transform* is based upon its definition while for the *inverse transform* usually indirect methods are used based upon the partial fraction expansion and polynomial division. These techniques may be simplified by various computer routines including MATLAB functions as well.

Application of Z-transform covers various possibilities of system description including *discrete transfer function* and *frequency response function* using the complex plane representation. Z-transform is often used to simplify the solution of difference equations, too.

Discrete Fourier transform closely related to Z-transform is a basic mathematical tool for signal decomposition and reconstruction. Its applications cover many engineering disciplines as well.

Various learning and adaptive discrete systems are based upon the use of the *least square method* fundamental in many optimization problems. In many cases *gradient methods* are applied.

4

Signal Analysis

Signal analysis is a very useful tool enabling to distinguish spectral components of the observed sequence of values. In some applications such an information is sufficient but in other cases it forms the basis for further processing only. The following chapter presents some methods and algorithms to achieve such an analysis of one or more observed signals based upon the discrete Fourier transform while parametric methods closely connected with random signal theory and signal modelling are discussed later.

4.1 Space-Frequency Analysis

4.1.1 Basic Spectral Estimation

Let us assume at first the problem of the infinite data length analysis. Instead of the discrete Fourier transform discussed before *discrete-time Fourier transform* applied for the infinite sequence $\{x_d(n)\}$ can be used based upon relation [15, p.372]

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x_d(n) e^{-j\omega n} \tag{4.1}$$

for continuous frequency ω with its period 2π long. The inverse discrete-time Fourier transform is then defined by relation

$$x_d(n) = \frac{1}{2\pi} \int_0^{2\pi} X_d(\omega) e^{j\omega n} d\omega \tag{4.2}$$

The last expression explains how the given signal is represented by the integral of its harmonic components.

Definition 4.1 *The magnitude and phase spectrum of a sequence* $\{x_d(n)\}_{n=-\infty}^{\infty}$ *is defined as the magnitude and phase of its discrete-time Fourier transform.*

Instead of spectrum sometimes *power spectral density* [22, p.59] is used defined by relation

$$S_{xx}(\omega) = \lim_{L\to\infty} E\left[\frac{1}{2L+1}\left|\sum_{n=-L}^{L} x_d(n) e^{-j\omega n}\right|^2\right] \tag{4.3}$$

with symbol E standing for the mean value of several realizations.

In real cases finite length data segment can be processed only which implies that

- a window function must be used to choose a sequence N samples long
- the discrete Fourier transform can be applied

The main problem of such an approximation is in application of a method which would provide the best estimate of the true function defined for the infinite process realization.

4.1.2 Window Functions

Various window sequences $\{w(n)\}$ of the finite length N are used to derive a finite sequence of the same length from the infinite signal $\{x_d(n)\}$ or impulse response $\{h_d(n)\}$ after their scalar multiplication. Two main applications of windowing include

- the choise of the data segment
$$x(n) = x_d(n)\, w(n) \tag{4.4}$$
to enable the signal analysis
- the evaluation of the finite impulse response filter defined by its impulse response
$$h(n) = h_d(n)\, w(n) \tag{4.5}$$
based on the ideal infinite impulse response.

We shall study now the application of windowing for spectral analysis while its use for filtering will be discussed later.

Using properties of the discrete-time Fourier transform [15, p.379] the scalar multiplication in Eq.(4.4) in time domain is equivalent to the periodic convolution in frequency domain given by relation

$$X(\omega) = \frac{1}{2\pi} X_d(\omega) \otimes W(\omega) \tag{4.6}$$

defined by integral

$$X(\omega) = \frac{1}{2\pi} \int_0^{2\pi} X_d(\Omega) W(\omega - \Omega) d\Omega \tag{4.7}$$

Theorem 4.1 *The discrete-time Fourier transform of sequence*

$$x_d(n) = \sum_i c_i e^{j\omega_i n} \tag{4.8}$$

is given by the linear combination of unit impulses in the form

$$X_d(\omega) = 2\pi \sum_i c_i \delta(\omega - \omega_i) \tag{4.9}$$

Proof: Applying Eq.(4.9) in Eq.(4.2) we obtain

$$x_d(n) = \frac{1}{2\pi} \int_0^{2\pi} 2\pi \sum_i c_i \delta(\omega - \omega_i) e^{j\omega n} d\omega$$

which can be simplified to relation

$$x_d(n) = \sum_i c_i \int_0^{2\pi} \delta(\omega - \omega_i) e^{j\omega n} d\omega$$

providing result in form (4.8).

Example 4.1 *Compare the ideal spectrum of the harmonic sequence* $x_d(n) = \cos(\omega_0 n)$ *and its window approximation.*

Solution: As

$$x_d(n) = \cos(\omega_0 n) = \frac{1}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n}) \tag{4.10}$$

it is possible to apply Theorem 4.1 to evaluate

$$X_d(\omega) = \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)) \tag{4.11}$$

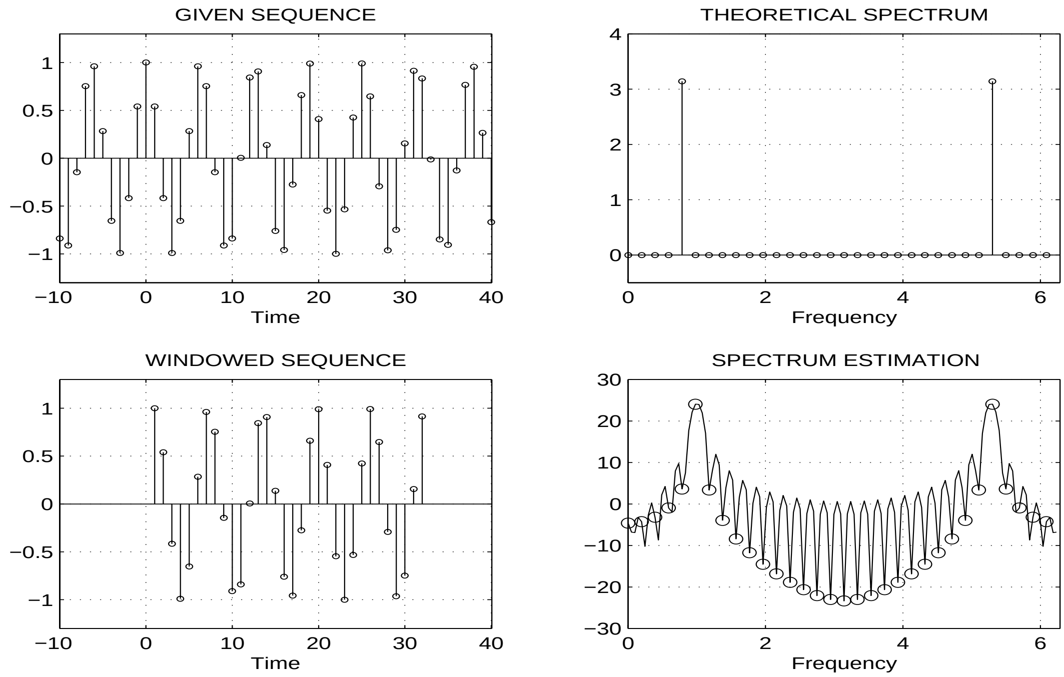

FIGURE 4.1. Theoretical harmonic infinite sequence of frequency $\omega_0 = 1$ with its line spectrum and the comparison with the finite sequence of length $N = 32$ values and its spectrum based upon the discrete-time and discrete Fourier transform.

Using further Eq. (4.6) we can estimate the window spectrum in the form

$$X(\omega) = \frac{1}{2}(W(\omega - \omega_0) + W(\omega + \omega_0)) \tag{4.12}$$

Properties of such a function are in this way entirely dependent upon the window function $\{w(n)\}$. Using the rectangular window defined below it is possible to evaluate the spectrum estimation given in Fig. 4.1.

Definitions of basic window function are summarized in Tab. 4.1 with their sketch and magnitude spectra in Fig. 4.2 based upon Algorithm 4.1. Ideal line spectra of harmonic

Window definition (MATLAB notation)	*Mainlobe width*	*Sidelobe level*
Rectangular (BOXCAR)		
$w(n) = \begin{cases} 1 & \text{for } n = 0, 1, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$4\pi/N$	$-13\ dB$
Bartlett (Triangular - BARTLETT)		
$w(n) = \begin{cases} 2n/N - 1 & \text{for } n = 0, 1, \cdots, (N-1)/2 \\ 2 - 2n/(N-1) & \text{for } n = (N-1)/2, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$8\pi/N$	$-27\ dB$
Hanning (HANNING)		
$w(n) = \begin{cases} (1 - \cos(2\pi n/(N-1)))/2 & \text{for } n = 0, 1, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$8\pi/N$	$-32\ dB$
Hamming (HAMMING)		
$w(n) = \begin{cases} 0.54 - 0.46\cos(2\pi n/(N-1)) & \text{for } n = 0, 1, \cdots, N-1 \\ 0 & \text{elsewhere} \end{cases}$	$8\pi/N$	$-43\ dB$

TABLE 4.1. Basic window function definition

signal components are according to Eq. (4.12) represented by shifted window spectra (using periodic extension) given in Fig. 4.2. The *mainlobe width* presents the ability to distinguish two closely spaced harmonic components. The *sidelobe level* enables to estimate how small signal can be detected in presence of large ones.

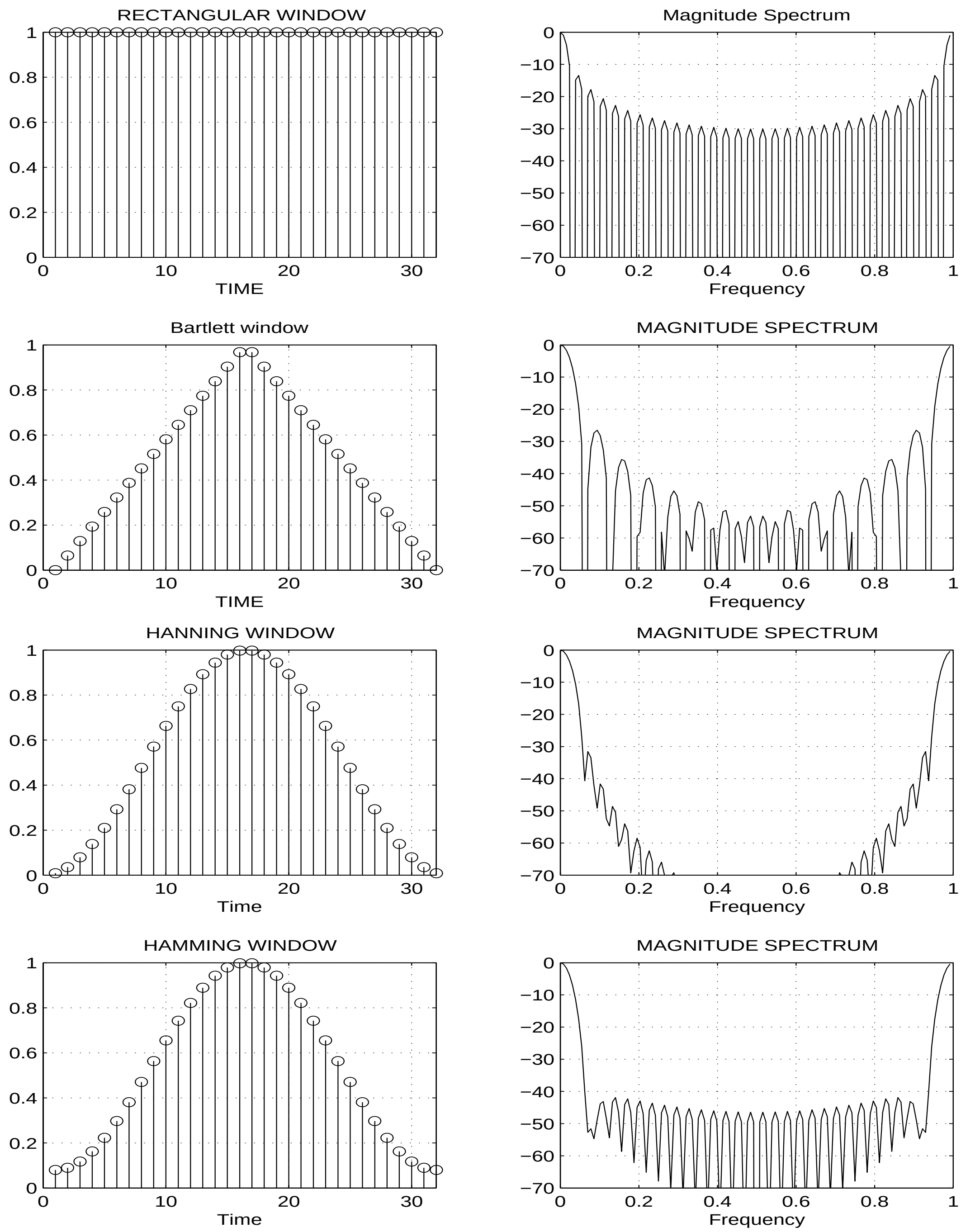

FIGURE 4.2. Basic window functions and their magnitude spectra in dB

Discrete-time Fourier transform enables to evaluate the discrete Fourier transform of a *finite length sequence* by sampling frequency ω over one period in N points. *Spectrum* can be in this way approximated by relation

$$X(k) = X(\omega) \mid_{\omega=k\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x(n) e^{-jk\frac{2\pi}{N}n} \tag{4.13}$$

Algorithm 4.1 *Basic window functions magnitude spectra evaluation.*

- matrix of window functions definitions for a given length N and their plot

  $\mathbf{w}$ = [boxcar(N) bartlett(N) hanning(N) hamming(N)];
  plot($\mathbf{w}$);

- magnitude spectra evaluation of a chosen length M

  $\mathbf{W}$ =fft($\mathbf{w}, M$);
  $\mathbf{f}$ = $[0 : (M-1)]/M$;
  semilogy($\mathbf{f}$,abs($\mathbf{W}$));

for $k = 0, 1, \cdots, N-1$. The power spectral density function defined by (4.3) can be similarly approximated by *periodogram* defined as

$$S_{xx}(k) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-jk\frac{2\pi}{N}n}\right|^2 \tag{4.14}$$

for $k = 0, 1, \cdots, N-1$ taking into account one realization only. Its properties and problems of its application are discussed in [22, p.68].

It is obvious that the spectrum or periodogram estimation based on Eqs. (4.13) and (4.14) can be evaluated using the MATLAB function *fft* mentioned in the previous section.

4.1.3 Short-Time Fourier Transform

In some applications of signal processing with its frequency components changing itself in time it is useful to combine time-domain and frequency domain analysis. In particular it is possible to evaluate the Fourier transform for each time in the neighborhood of that instant [22].

Using the shifted window function $w(m)$ in Eq.(4.4) by n samples and applying Eq. (4.1) we can define the discrete short-time Fourier transform (STFT) by relation

$$X(n,\omega) = \sum_{m=-\infty}^{\infty} x(m)w(n-m)e^{-j\omega m} \tag{4.15}$$

for $n = 0, 1, \cdots$. allowing to choose a short-time section of $\{x(m)\}$ at time n. As N samples long window function implies N samples long time section processing the discrete Fourier transform can be used resulting according to Eq. (4.13) in relation

$$X(n,k) = \sum_{m=n-(N-1)}^{n} X(m)w(n-m)e^{-jk\frac{2\pi}{N}m} \tag{4.16}$$

for any time index n and $k = 0, 1, \cdots, N-1$.

Example 4.2 *Let us apply the short-time Fourier transform to a sequence*

$$x(n) = \sin((\omega_0\, n)n)$$

for $\omega_0 = 0.07$ *and* $n = 0, 1, \cdots, 100$ *using* $N = 32$ *samples long rectangular window function.*

Solution: Applying Eq. (4.16) it is possible to evaluate result given in Fig. 4.3 presenting time dependent signal frequency.

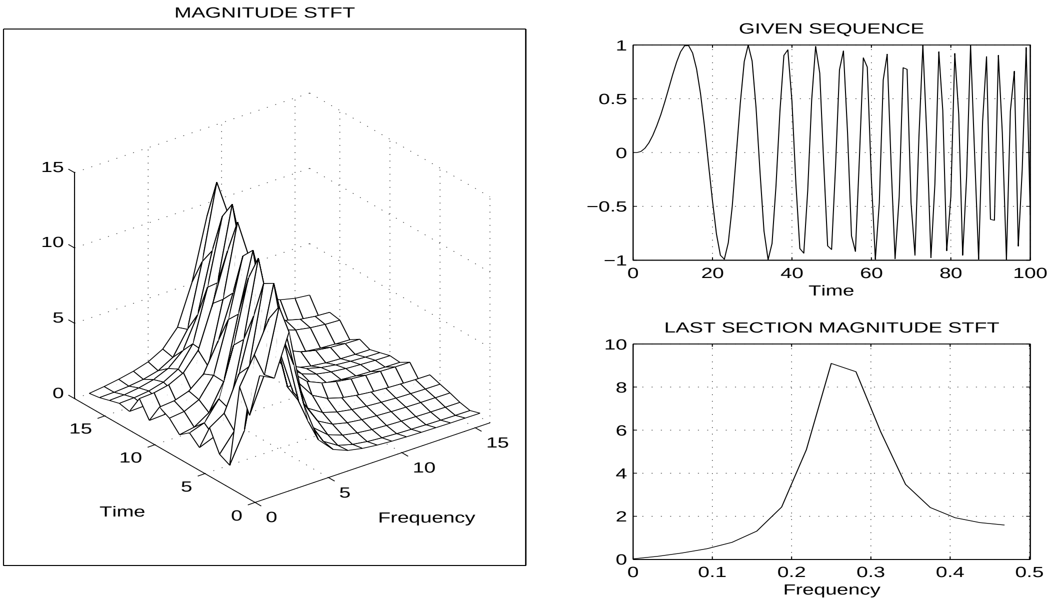

FIGURE 4.3. The short time Fourier transform of sequence $x(n) = \sin((\omega_0\, n)n)$ for $\omega_0 = 0.01$ using rectangular window function $w(n)$ of length $N = 32$ samples and $n = 32, 36, 38, \cdots, 100$.

4.1.4 Cepstral Analysis

Many signal processing methods are based upon presumption that the given system is linear. Such a system is relatively easy to analyse and it is possible to design a variety of useful signal processing functions.

Let us consider now a special class of nonlinear systems with their system transformation H that obey a generalized principle of superposition discussed in more general way in [30] and given by the following relation

$$H[x_1(n)\,(op1)\,x_2(n)] = H[x_1(n)]\,(op1)\,H[x_2(n)] \tag{4.17}$$

$$H[c\,(op2)\,x_1(n)] = c\,(op2)\,H[x_1(n)] \tag{4.18}$$

where $(op1)$ and $(op2)$ stand for any operators satisfying the algebraic postulates of vector addition and scalar multiplication. Such *homomorphic systems* include linear systems as well.

In practical cases it is possible to restrict our interest to systems combining their signals either by multiplication or convolution. Especially the last possibility is studied very often as it can be used for a general problem of echo detection to improve the quality of accoustic signals or analyze behaviour of seismic signals etc.

The main principle of the homomorphic system processing include

- application of the direct characteristic system transforming operator $(op1)$ to addition for signal $\{x(n)\}$ to evaluate $\{c(n)\}$
- processing of signal $\{c(n)\}$ by any linear method
- application of the inverse characteristic system transforming operator of addition back to $(op1)$.

This principle is presented in Fig. 4.4 for systems combining their signals by *convolution*. The *direct characteristic system* is based upon the discrete Fourier transform changing convolution to multiplication, following application of the logarithmic function combining

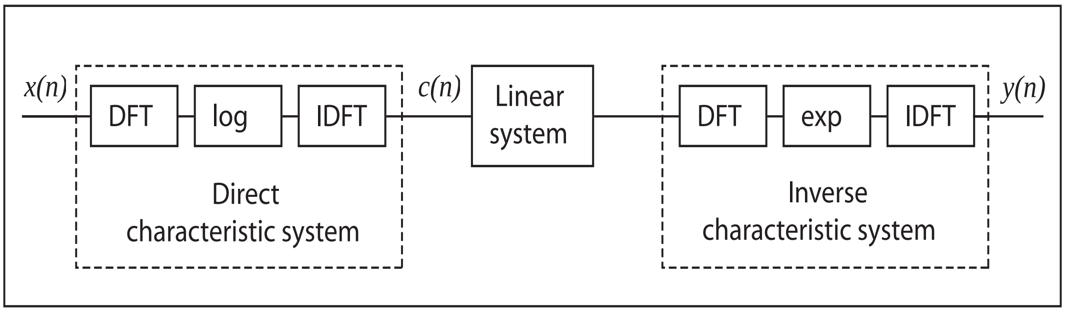

FIGURE 4.4. Representation of a homomorphic system with a complex cepstrum as an output of the direct characteristic system.

signals by addition and inverse Fourier transform returning further processing to the time domain. Resulting signal is called *complex cepstrum* allowing to detect signal echos and their cancelling by a linear system. *Inverse characteristic system* is similar to the direct one with complex exponential instead of logarithm only.

Computer processing of signal $\{x(n)\}$ to obtain its cepstrum is presented in Algorithm 4.2. The whole evaluation can be achieved by the MATLAB function *cceps* as well.

Example 4.3 *Let us analyse cepstrum of signal*

$$x(n) = r(n) \otimes s(n) = \sum_{m=0}^{N-1} r(m)s(n-m)$$

representing given signal and its echo in the form

$$x(n) = r(n) + \alpha r(n - n_1)$$

where

$$\begin{aligned} r(n) &= e^{-0.1n} \sin(\frac{\pi}{4} n) \\ s(n) &= \delta(n) + \alpha\delta(n - n_1) \end{aligned}$$

with $\delta(n)$ standing for an impulse function, $n_1 = 40$ and $\alpha = 0.5$.

Solution: The direct characteristic system involves

- the discrete Fourier transform application

$$X(k) = R(k)S(k)$$

where

$$\begin{aligned} R(k) &= \sum_{n=0}^{N-1} r(n)e^{-jk\frac{2\pi}{N}n} \\ S(k) &= \sum_{n=0}^{N-1} s(n)e^{-jk\frac{2\pi}{N}n} = 1 + \alpha e^{-jk\frac{2\pi}{N}n_1} \end{aligned}$$

- the logarithmic function application

$$\begin{aligned} L(k) &= \log(X(k)) = \log|X(k)| + j\arg(X(k)) = \\ &= \log|R(k)| + \log|S(k)| + j(\arg(R(k) + \arg(S(k))) \end{aligned}$$

The contribution to the complex logarithm due to the impulse train for $|\alpha| < 1$ is

$$L^s(k) = \log\left(1 + \alpha e^{-jk\frac{2\pi}{N}n_1}\right) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i} \left(\alpha e^{-jk\frac{2\pi}{N}n_1}\right)^i$$

Algorithm 4.2 *Cepstral analysis of a given signal* $\{x(n)\}$, $n = 0, 1, \cdots, N-1$ *assuming original signals in convolution.*

- discrete Fourier transform application
  $\mathbf{X}$ =fft($\mathbf{x}$);
- complex logarithm implementation
  $\mathbf{L}$ =log($\mathbf{X}$);
- inverse Fourier transform application
  $\mathbf{c}$ =ifft($\mathbf{L}$);

- the inverse discrete Fourier transform use which for the signal component $s(n)$ provides the contribution to the complex cepstrum in the form

$$c^s(n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}\alpha^i e^{-jk(n_1 i-n)\frac{2\pi}{N}} = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{\alpha_i}{i}\delta(n-n_1 i)$$

  As this sequence is additive to the sequence given by the original signal it is obvious that cepstrum can be used for the echo analysis. Results are presented in Fig. 4.5.

Cepstrum component due to the echo can be usually eliminated by a simple window function in the form

$$w(n) = \begin{cases} 0 & \text{for } n < n_i \\ 1 & \text{for } n > n_c \end{cases} \tag{4.19}$$

where n_c stands for the index resulting from the cepstral analysis. Inverse characteristic system can be then used. Results of such a procedure applied to the example given above are presented in Fig. 4.5 as well.

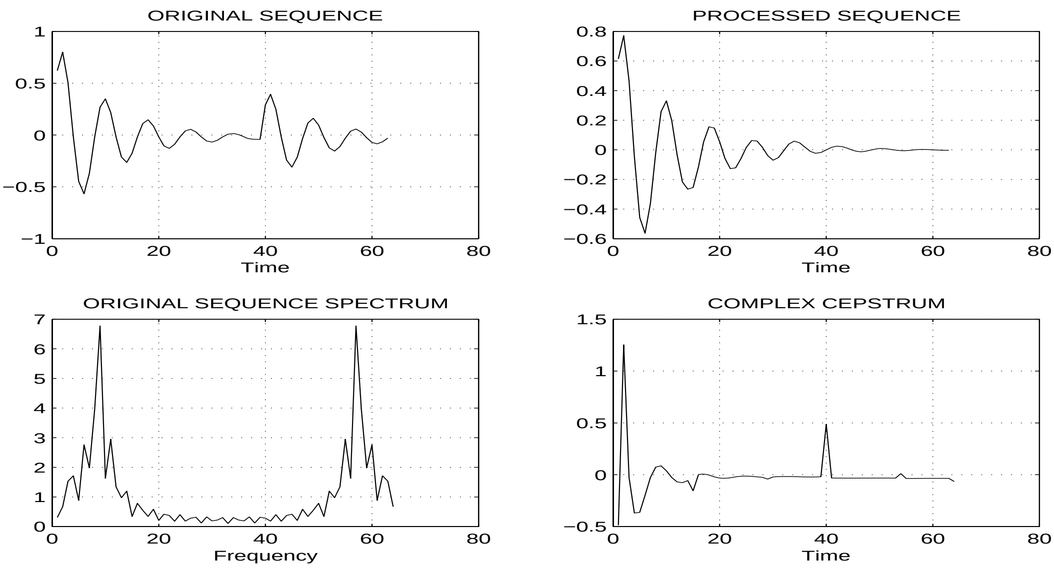

FIGURE 4.5. Original sequence $\{x(n)\}_{n=0}^{N-1}$ composed of sequence $r(n) = e^{-0.1n}\sin(\frac{\pi}{4}n)$ and its echo delayed by $n_1 = 40$ samples attenuated by coefficient $\alpha = 0.5$ for $N = 64$ before and after the homomorphic processing using the window function for its cutoff index $n_c = 30$ and the original sequence magnitude spectrum and complex cepstrum

4.1.5 Two-Dimensional Signal Analysis

While one-dimensional signal analysis and processing can be applied in many engineering and scientific systems producing sequences of values its two-dimensional extension is mainly used in video processing. The mathematical background of such an analysis is based upon discrete space signals $\{x(n_1, n_2)\}$ defined for all integer values of n_1 and n_2. The most important deterministic signals represent

- unit sample sequence: $\delta(n_1, n_2) = \begin{cases} 1 & \text{for } n_1 = n_2 = 0 \\ 0 & \text{otherwise} \end{cases}$
- unit step sequence: $u(n_1, n_2) = \begin{cases} 1 & \text{for } n_1, n_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- real exponential sequence: $x(n_1, n_2) = A\alpha^{n_1}\beta^{n_2}$
- rectangular sequence: $x(n_1, n_2) = \begin{cases} 1 & \text{for } a \leq n_1 \leq b, c \leq n_2 \leq d \\ 0 & \text{otherwise} \end{cases}$

Sketch of signals is given in Fig. 4.6.

It is possible to show [22] that any stable sequence $\{x(n_1, n_2)\}$ can be defined by combination of complex exponentials with coefficients $X(\omega_1, \omega_2)$ according to the following definition.

Definition 4.2 *The direct discrete space Fourier transform is given by relation*

$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \tag{4.20}$$

while the inverse discrete space Fourier transform is defined as

$$x(n_1, n_2) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2 \tag{4.21}$$

Result of the direct *discrete space Fourier transform* is in general a complex function of continuous variables ω_1 and ω_2 having period 2π which implies that

$$X(\omega_1, \omega_2) = X(\omega_1 + 2\pi, \omega_2) = X(\omega_1, \omega_2 + 2\pi) \tag{4.22}$$

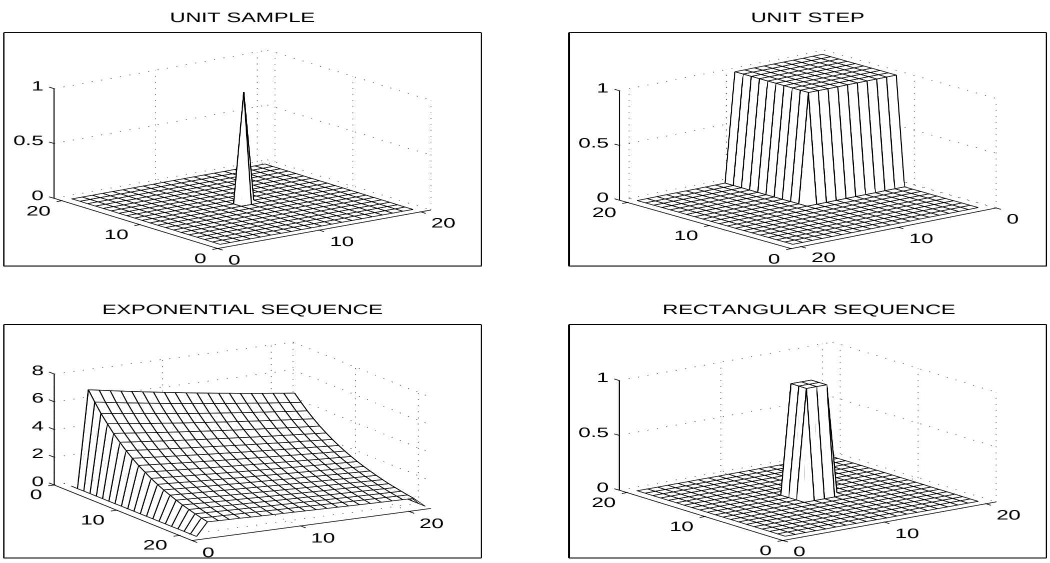

FIGURE 4.6. Basic deterministic two-dimensional signals.

and it represents the signal *spectrum.* In case that $\{x(n_1, n_2)\}$ stands for the two-dimensional linear system impulse response its discrete space Fourier transform represents the *frequency response* of the system.

In many cases the given sequence $\{x(n_1, n_2)\}$ has a finite length with its indices in the range $0 \le n_1 \le N_1 - 1$ and $0 \le n_2 \le N_2 - 1$. The *discrete Fourier transform* $X(k_1, k_2)$ of such a sequence is related to the discrete space Fourier transform by relation

$$X(k_1, k_2) = X(\omega_1, \omega_2)\,|_{\omega_1 = k_1 \frac{2\pi}{N_1},\, \omega_2 = k_2 \frac{2\pi}{N_2}} \tag{4.23}$$

for $0 \le k_1 \le N_1 - 1$, $0 \le k_2 \le N_2 - 1$ and it can be evaluated by the following definition.

Definition 4.3 *The direct discrete Fourier transform is given by relation*

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-jk_1 \frac{2\pi}{N_1}} e^{-jk_2 \frac{2\pi}{N_2}} \tag{4.24}$$

for $0 \le k_1 \le N_1 - 1$, $0 \le k_2 \le N_2 - 1$ *and the inverse discrete Fourier transform is defined as*

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) e^{jk_1 \frac{2\pi}{N_1}} e^{jk_2 \frac{2\pi}{N_2}} \tag{4.25}$$

This definition implies that the sequence $\{x(n_1, n_2)\}$ of length $N_1\, N_2$ in space domain is represented by sequence $\{X(k_1, k_2)\}$ of the same size in frequency domain.

Example 4.4 *Evaluate the discrete-space Fourier transform of the rectangular sequence*

$$x(n_1, n_2) = \begin{cases} 1 & for\ -1 \le n_1 \le 1, -1 \le n_2 \le 1 \\ 0 & otherwise \end{cases} \tag{4.26}$$

given in Fig 4.6.

Solution: Using the Definition 4.2 it is possible to find

$$\begin{aligned} X(\omega_1, \omega_2) &= 1 + e^{j\omega_1} e^{-j\omega_2} + e^{-j\omega_2} + e^{-j\omega_1} e^{-j\omega_2} + e^{j\omega_2} + e^{-j\omega_1} + e^{j\omega_1} e^{j\omega_2} + e^{j\omega_2} + e^{-j\omega_1} e^{j\omega_2} = \\ &= 1 + 2(\cos(\omega_1 - \omega_2) + \cos(\omega_1) + \cos(\omega_2) + \cos(\omega_1 + \omega_2)) \end{aligned}$$

Sketch of this (real) function is presented in Fig. 4.7.

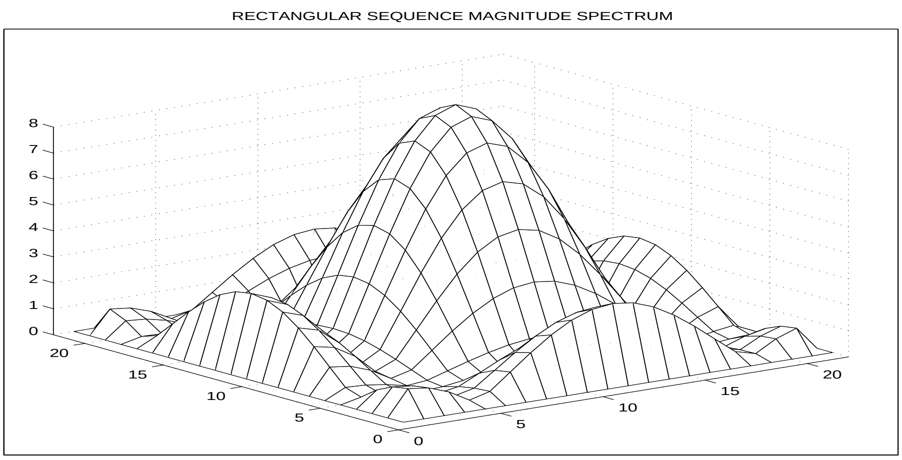

FIGURE 4.7. The discrete space Fourier transform of the rectangular sequence

The application of the two-dimensional Fourier representation of a signal can be used both for signal analysis and its processing using similar principles applied for the one-dimensional case. Further discussion is given in [22].

4.2 Summary

Signal analysis is a very powerful tool to provide spectral estimation of signal components. This chapter presented basic ideas only connected with the discrete Fourier transform and applications of window functions. Further information will be provided in next sections.

5

Digital Filters

Any digital systems enabling transformation of a given sequence and using an algorithmic mathematical approach to such a procedure are often called *digital filters*. Applications of these structures cover many areas including system identification and modelling, signal detection, interference cancelling and system control as well. The following section is devoted to basic nonadaptive filters while adaptive systems will be discussed further.

5.1 Basic Principles and Methods

A very extensive theory of analog filters has been developed before digital systems started to be so widely used. It is one of reasons why for many applications prototype analog filters are designed at first and then transformed to their digital form [5, 23] enabling their more precise and simple implementation. As such an approach has been described in many references till now we shall concentrate our attention to the direct digital system design in most cases.

5.1.1 Digital System Description

A general linear shift invariant discrete system can be described by the difference equation

$$y(n) + \sum_{k=1}^{N} a(k)y(n-k) = \sum_{k=0}^{N} b(k)x(n-k) \tag{5.1}$$

or according to the mathematical analysis given above by its discrete transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1 + \sum_{k=1}^{N} a(k)z^{-k}} \tag{5.2}$$

or frequency response

$$H(e^{j\omega_k}) = H(z) \mid_{z=e^{j\omega_k}} \tag{5.3}$$

While the difference equation is essential for transformation of signal $\{x(n)\}$ to $\{y(n)\}$ the frequency response function enables the analysis and design of the discrete system behaviour describing which frequency components of a given signal will be rejected. Basic ideal frequency responses of various filters are summarised in Fig. 5.1 for digital frequency $\omega \in \langle 0, \pi)$. Filter design for approximation of such ideal characteristics presented in Fig. 5.1 involves

- the estimation of the form of difference equation (5.1) resulting in *finite impulse response* (FIR) filter for zero coefficients $\{a(1), a(2), \cdots, a(N)\}$ or *infinite impulse response* (IIR) filter
- the evaluation of the difference equation coefficients

These steps are closely connected with demands covering the accuracy of the frequency response approximation, system stability and its behaviour.

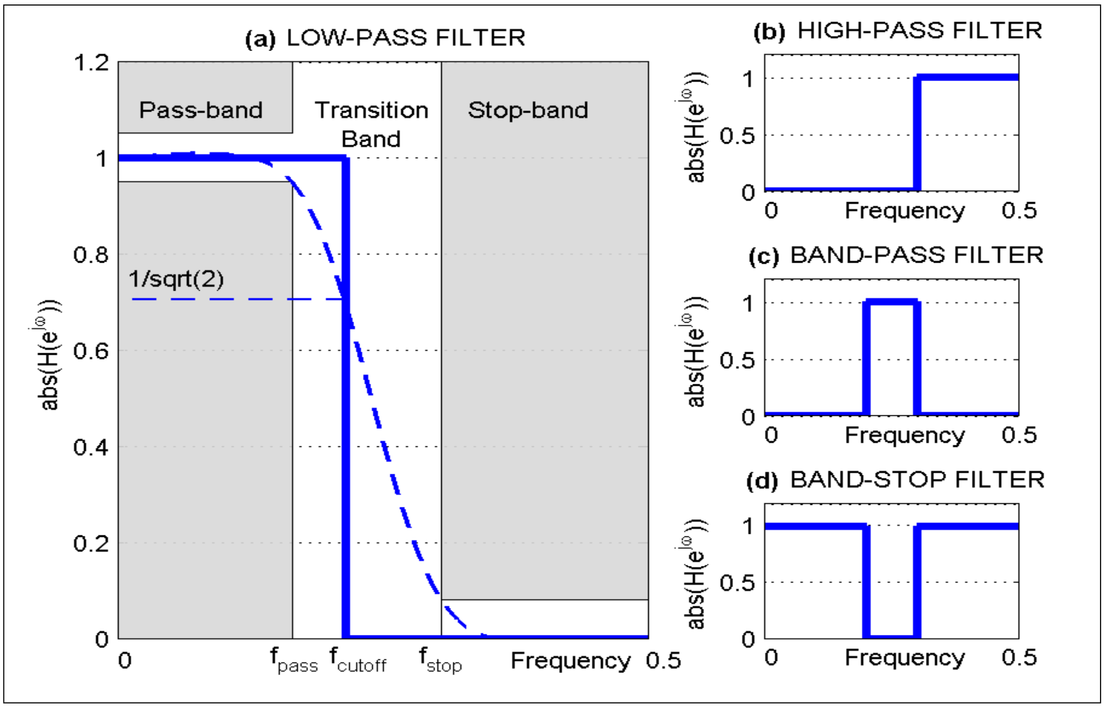

FIGURE 5.1. Amplitude frequency characteristics of basic types of filters including **(a)** ideal and real low-pass filter, **(b)** high-pass, **(c)** band-pass, and **(d)** band-stop filters

Example 5.1 *Use the discrete system described by equation*

$$y(n) - y(n-1) + 0.5\,y(n-2) = 0.2\,x(n-1) + 0.2\,x(n-2)$$

to evaluate its response to the input sequence

$$x(n) = \sin(0.1\,n) + \sin(2.5\,n)$$

Solution: According to results of Example 3.5 the transfer function of a given system in the form

$$H(z) = 0.2\frac{z+1}{z^2 - z + 0.5}$$

implies the amplitude frequency response $|H(e^{j\omega})|$ presented in Fig. 5.2 approximating the low-pass filter. As the spectrum of the given sequence has only one its component in the system pass-band the difference equation rejects the high frequency signal component with results given in Fig. 5.2.

5.1.2 Elementary Digital Filters

In various applications very simple digital filters can be used even though their frequency response is not quite well in comparison with the ideal one.

The MOVING AVERAGE system can be described by the *difference equation*

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(n-k) \tag{5.4}$$

taking into account N values of a given sequence. As

$$\begin{aligned} y(n) &= \frac{1}{N}(x(n) + x(n-1) + x(n-2) + \cdots + x(n-N+2) + x(n-N+1)) \\ y(n-1) &= \frac{1}{N}(x(n-1) + x(n-2) + x(n-3) + \cdots + x(n-N+1) + x(n-N)) \end{aligned}$$

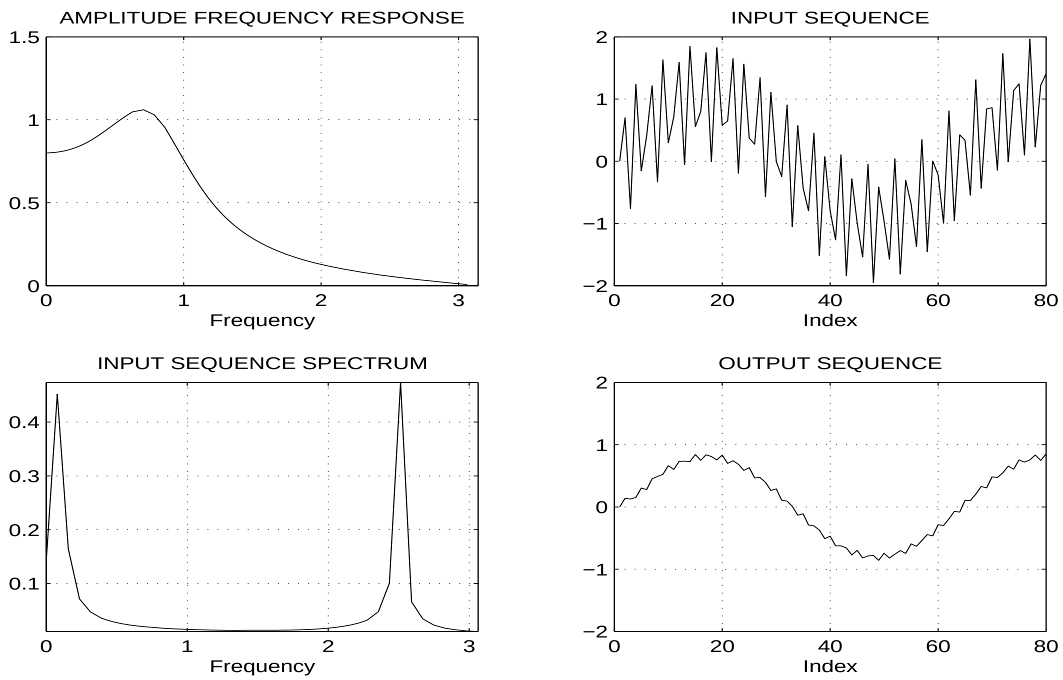

FIGURE 5.2. Amplitude frequency response of the discrete system with the transfer function $H(z) = 0.2(z+1)/(z^2 - z + 0.5)$ and results of its application for processing of sequence $x(n) = \sin(0.1\,n) + \sin(2.5\,n)$

which implies that

$$y(n) - y(n-1) = \frac{1}{N}\,(x(n) - x(n-N))$$

the discrete *transfer function* is in the form

$$H(z) = \frac{1}{N}\,\frac{z^N - 1}{z^N - z^{N-1}} \tag{5.5}$$

and the *frequency response* can be expressed after a few mathematical operations (presented in Example 2.3 in the form

$$H(e^{j\omega}) = \frac{1}{N} e^{-j\omega(N-1)/2} \frac{\sin \omega N/2}{\sin \omega/2} \tag{5.6}$$

with

$$\left|H(e^{j\omega})\right| = \frac{1}{N}\left|\frac{\sin \omega N/2}{\sin \omega/2}\right| \tag{5.7}$$

Graphic interpretation of this result is presented in Fig. 5.3 for $N = 10$ showing that the moving average approximates the low-pass filter with N allowing to change its cutoff frequency.

The EXPONENTIAL DECAY system can be described in a similar way by the *difference equation*

$$y(n) = \frac{1}{\sum_{k=0}^{N-1} v(k)} \sum_{k=0}^{N-1} v(k) x(n-k) \tag{5.8}$$

for $v(k) = v(1)^k$ and $v(1) \in \langle 0, 1 \rangle$ which for $v(1) = 1$ stands for the moving average in fact. As

$$y(n) - v(1)\,y(n-1) = \frac{1}{\sum_{k=0}^{N-1} v(k)}\,(x(n) - v(N)\,x(n-N))$$

the *discrete transfer function* can be expressed in the form

$$H(z) = \frac{1}{\sum_{k=0}^{N-1} v_k}\,\frac{z^N - v(N)}{z^N - v(1) z^{N-1}} \tag{5.9}$$

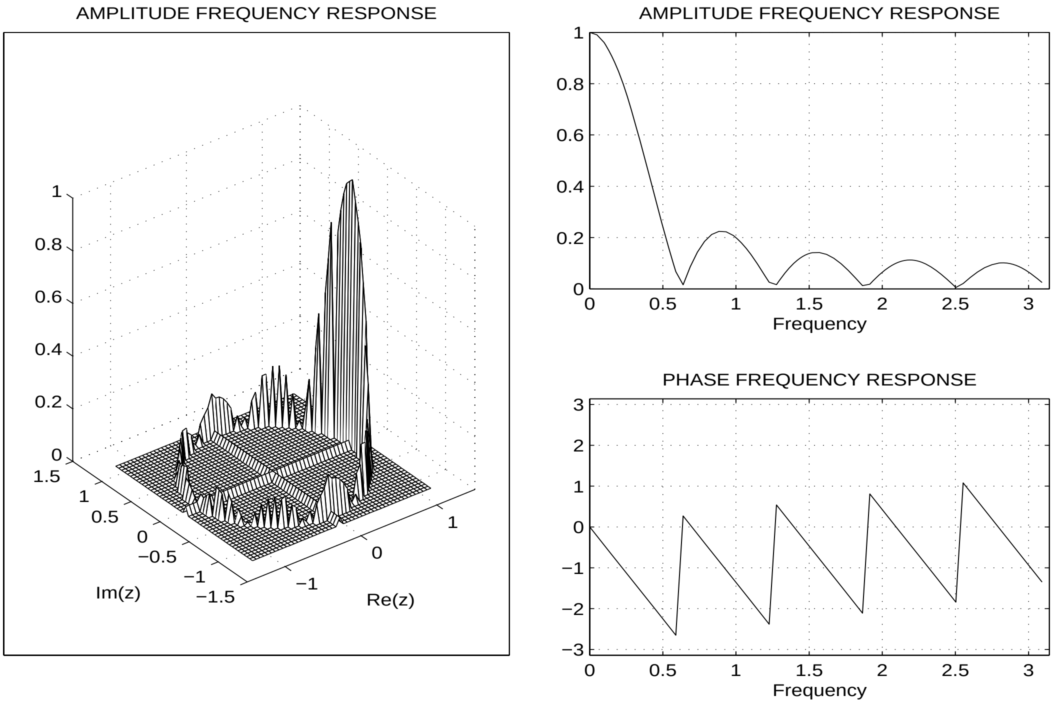

FIGURE 5.3. Magnitude and phase frequency response of the moving average system with the transfer function $H(z) = (z^N - 1)/(N\,(z^N - z^{N-1}))$ for $N = 10$ and its sketch in the complex plane

System amplitude *frequency response* for $v(N) << 1$ in form

$$\left|H(e^{j\omega})\right| \approx \frac{1 - v(1)}{\sqrt{1 + v(1)^2 - 2v(1)\cos\omega}} \tag{5.10}$$

presented in Fig. 5.4 represents the approximation of the low-pass filter again allowing to use coefficient $v(1)$ to change its cutoff frequency.

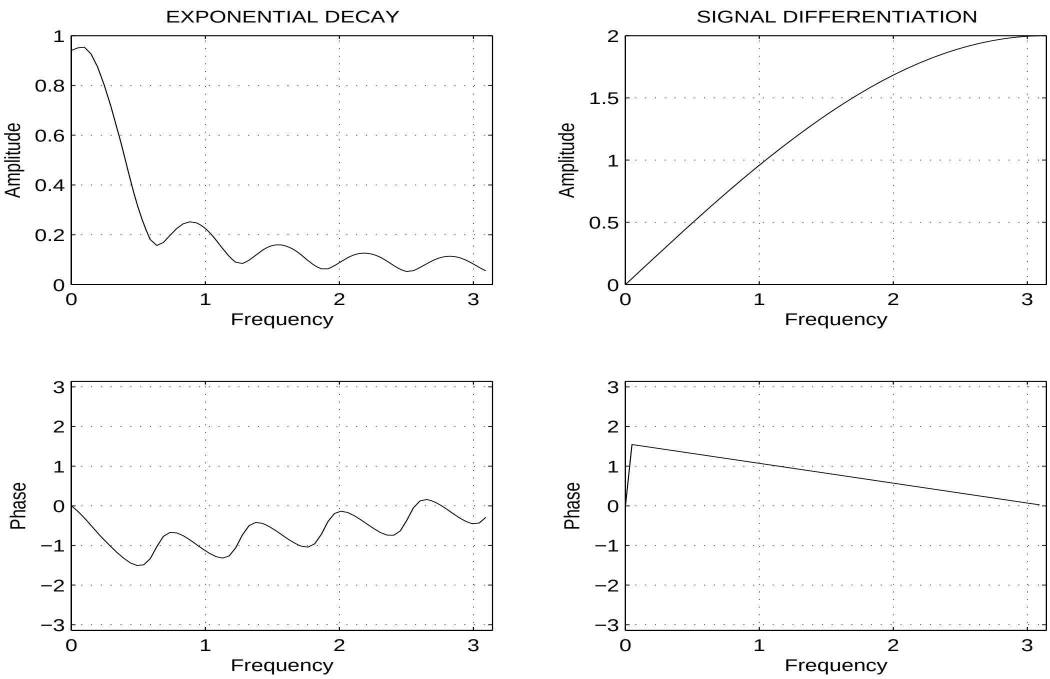

FIGURE 5.4. Magnitude and phase frequency response of the exponential decay system and the digital system for signal differentiation.

The SIGNAL DIFFERENTIATION described by the *difference equation*

$$y(n) = x(n) - x(n-1) \tag{5.11}$$

and the *discrete transfer function*

$$H(z) = 1 - z^{-1} \tag{5.12}$$

has its *frequency response* in the form

$$H(e^{j\omega}) = 1 - e^{-j\omega} \tag{5.13}$$

Its magnitude

$$|H(e^{j\omega})| = \sqrt{(1-\cos\omega)^2 - \sin^2\omega} = 2\sin(\omega/2) \tag{5.14}$$

presented in Fig. 5.4 represents a very simple approximation of the high-pass filter with no possiblity of its cutoff frequency change.

5.2 Finite Impulse Response Filter

We shall study now the digital system described by the difference equation

$$y(n) = \sum_{k=0}^{P-1} h(k)\, x(n-k) \tag{5.15}$$

derived from Eq. (5.1) which represents the general moving average system in fact. The purpose of the following analysis will be in the estimation of P and evaluation of values $\{h(0), h(1), \cdots, h(P-1)\}$ standing for the impulse response approximating the ideal frequency characteristics.

5.2.1 Linear Phase Systems

Definition 5.1 *A system with its finite impulse response $\{h(0), h(1), \cdots, h(P-1)\}$ is called a linear phase system if its frequency characteristics can be expressed in the form*

$$H(e^{j\omega_k}) = \sum_{n=0}^{P-1} h(n) e^{-jn\omega_k} = \left|H(e^{j\omega_k})\right| e^{-j\alpha\omega_k} \tag{5.16}$$

for $\omega_k = k\frac{2\pi}{P}, k = 0, 1, \cdots, P-1$.

Systems of this type [23] have an essential role in signal processing applications as they cause the same delay for all signal frequency components in their passband and preserve the shape of a given signal. In case that $DFT[x(n)]$ stands for the discrete Fourier transform of the input sequence the discrete Fourier transform of the system output can be evaluated in the form

$$DFT[y(n)] = \left|H(e^{j\omega})\right| e^{-j\alpha\omega} DFT[x(n)] = \left|H(e^{j\omega})\right| DFT[x(n-\alpha)] \tag{5.17}$$

It is obvious that if all frequency components of the input signal are in the passband of the system then $y(n) = x(n-\alpha)$ and the signal is delayed only.

Theorem 5.1 *A finite impulse response system of length P is a linear phase system in case that*

$$h(n) = h(P-1-n) \tag{5.18}$$

for $n = 0, 1, \cdots, P-1$.

Proof: In case of even P it is possible to use the definition of the discrete Fourier transform to find

$$H(e^{j\omega_k} = \sum_{n=0}^{P-1} h(n)e^{-j\omega_k n} = \sum_{n=0}^{P/2-1} h(n)e^{-j\omega_k n} + \sum_{n=P/2}^{P-1} h(n)e^{-j\omega_k n}$$

Using further Eq. (5.18) we can evaluate

$$\begin{aligned} H(e^{j\omega_k}) &= \sum_{n=0}^{P/2-1} h(n)e^{-j\omega_k n} + \sum_{n=0}^{P/2-1} h(n)e^{-j\omega_k(-n+P-1)} = \\ &= e^{-j\omega_k(P-1)/2} \sum_{n=0}^{P/2-1} h(n)(e^{-j\omega_k(n-(P-1)/2)} + e^{-j\omega_k(-n+(P-1)/2)}) = \\ &= e^{-j\omega_k(P-1)/2} \sum_{n=0}^{P/2-1} 2h(n)\cos(\omega_k(n-\frac{P-1}{2})) \end{aligned}$$

Comparison of the last expression with Eq. (5.16) provides value of

$$\alpha = (P-1)/2 \tag{5.19}$$

Similar results can be obtained for odd P.

5.2.2 *Ideal Frequency Response Approximation*

Let us assume an ideal low-pass filter according to Fig. 5.1(a) with its linear phase frequency response in the form

$$H(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & \text{for } 0 < \omega < \omega_c \text{ and } 2\pi - \omega_c < \omega < 2\pi \\ 0 & \text{for } \omega_c \le \omega \le 2\pi - \omega_c \end{cases} \tag{5.20}$$

with discrete values of $\omega_k = k\ 2\pi/P$ approaching continuous variable ω for $P \to \infty$. For this conditions discrete Fourier transform is usually called Fourier transform only and we can apply it for the approximation of the periodic extension of function (5.20) in the form

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-jn\omega} \tag{5.21}$$

where

$$h(n) = \frac{1}{2\pi}\int_0^{2\pi} H(e^{j\omega})e^{jn\omega}d\omega \tag{5.22}$$

It is further possible to evaluate

$$\begin{aligned} h(n) &= \frac{1}{2\pi}\int_0^{\omega_c} e^{-j(n-\alpha)\omega}d\omega + \frac{1}{2\pi}\int_{2\pi-\omega_c}^{\omega_c} e^{j(n-\alpha)\omega}d\omega = \\ &= \frac{1}{2\pi}\left[\frac{e^{j(n-\alpha)\omega}}{j(n-\alpha)}\right]_0^{\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j(n-\alpha)\omega}}{j(n-\alpha)}\right]_{2\pi-\omega_c}^{2\pi} = \\ &= \frac{1}{2\pi j(n-\alpha)}(e^{j(n-\alpha)\omega} - 1 + 1 - e^{-j(n-\alpha)\omega}) = \frac{\sin((n-\alpha)\omega)}{\pi(n-\alpha)} \end{aligned}$$

Using a finite sequence $\{h(n)\}$ only limited to P values it is possible to use Eq. (5.19) to define the linear phase impulse response in the form

$$h(n) = \frac{\sin((n-(P-1)/2)\omega_c)}{\pi(n-(P-1)/2)} \tag{5.23}$$

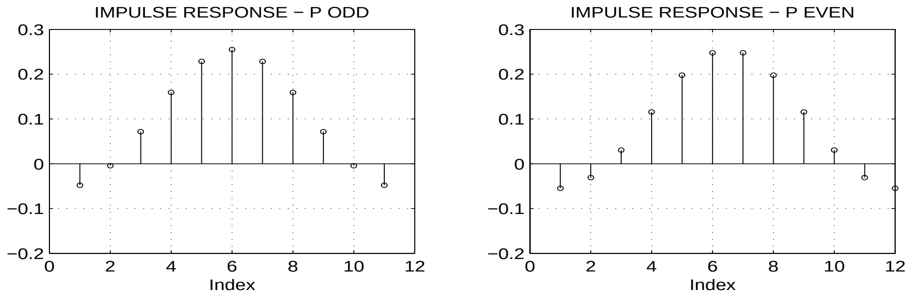

FIGURE 5.5. Finite impulse response $\{h(n)\}$ for P odd ($P = 11$) and even ($P = 12$)

for $n = 0, 1, \cdots, P-1$. The sketch of such an impulse response for odd and even P is presented in Fig. 5.5. The magnitude and phase frequency characteristics for a chosen length P and cutoff frequency is given in Fig. 5.6 confirming the assumption of the linear phase.

The process of the impulse response restriction to P values only can be looked upon as the scalar multiplication of the infinite impulse response by the rectangular window presented in the previous chapter and causing the transition band of length $4\pi/P$ [23, p.201]. The choice of P large enough can restrict the transition band under a given limit.

Comparison of finite impulse response filters for various length P is given in Fig. 5.7 presenting their amplitude frequency responses in dB defined as

$$H(e^{j\omega_k}) \mid_{dB} = 20 \log(|H(e^{j\omega_k})|/|H(e^{j\omega_0})|) \tag{5.24}$$

with $\omega_0 = 0$ for the low-pass filter.

The design procedure of the finite impulse response filter is summarized in Algorithm 5.1 (with the choice of $OMS = 0$ for the low-pass system).

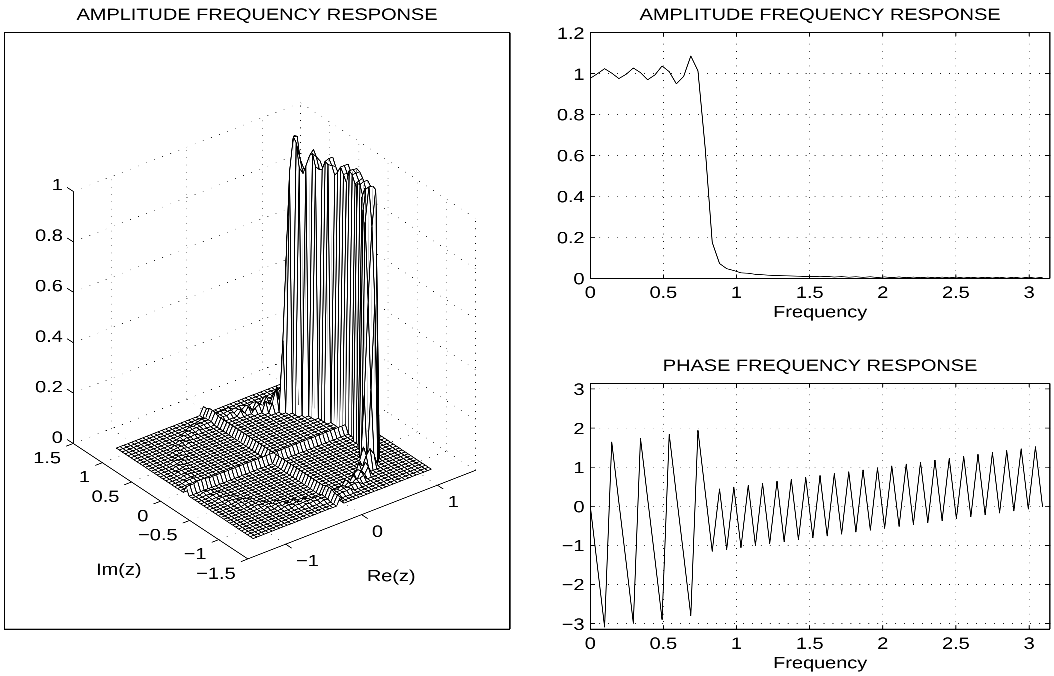

FIGURE 5.6. Magnitude and phase frequency characteristics of the finite impulse response filter of length $P = 64$ for cutoff frequency $\omega_c = 0.8$ [rad] and its sketch in the complex plane

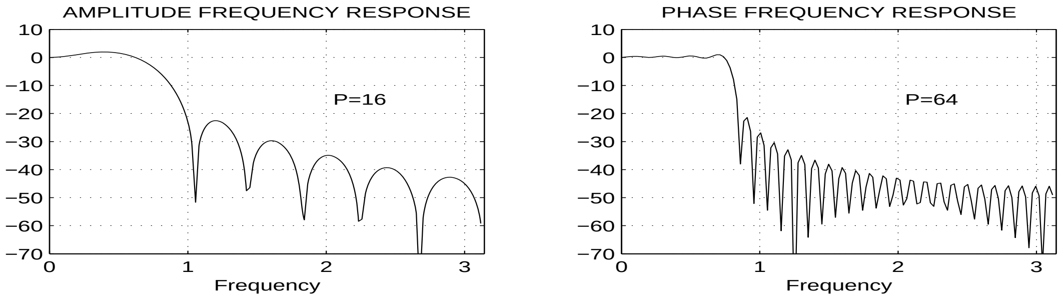

FIGURE 5.7. Low-pass amplitude frequency response of the FIR filter limited by the rectangular window of various length and cutoff frequency $\omega_c = 0.8$ [rad]

5.2.3 Basic Impulse Response Modifications

Finite impulse response filter of the low-pass type with its cutoff frequency ω_c derived above can be very simply modified to form a band-pass filter with its pass-band $2\omega_c$ long and the central frequency ω_s. Original impulse response $h(0), h(1), \cdots, h(P-1)$ implies the magnitude frequency response

$$\left|H(e^{j\omega_k})\right| = \left|\sum_{n=0}^{P-1} h(n)e^{-j\omega_k n}\right| \tag{5.25}$$

The shift by ω_s in frequency domain presented in Fig. 5.8 implies the band-pass magnitude frequency response in the form

$$\left|H(e^{j(\omega_k-\omega_s)})\right| = \left|\sum_{n=0}^{P-1} h(n)e^{-j(\omega_k-\omega_s)n}\right| = \left|\sum_{n=0}^{P-1} \tilde{h}(n)e^{-j\omega_k n}\right| \tag{5.26}$$

Algorithm 5.1 *Design of the low-pass and band-pass linear phase FIR filter.*

- choice of the filter length P, the cutoff frequency OMC of the original low-pass filter and central frequency OMS of the band-pass filter (enabling the low-pass filter design for $OMS = 0$).

- evaluation of the original impulse response values
  $\mathbf{p} = 0 : P-1;$
  $\mathbf{h} = \sin((\mathbf{p} - (P-1)/2) * OMC)./(\pi * (\mathbf{p} - (P-1)/2));$

- definition of the complex impulse response values of the band-pass filter
  $\mathbf{h} = \mathbf{h}. * \exp(j * OMS * \mathbf{p});$

- evaluation and plot of a given number of values M of the filter frequency response
  $[\mathbf{hh}, \mathbf{w}] = \text{freqz}\ (\mathbf{h}, [1, \text{zeros}(1, P-1)], M);$
  clg; subplot(211);
  plot($\mathbf{w}$, abs($\mathbf{hh}$)); plot($\mathbf{w}$, angle($\mathbf{hh}$));
  pause

- evaluation of the system output for a given sequence $\mathbf{x} = [x(1), x(2), \cdots, x(N)]$
  for $n = P : N$
  $y(n) = \mathbf{h} * \mathbf{x}(n : -1 : n-P+1)'$
  end

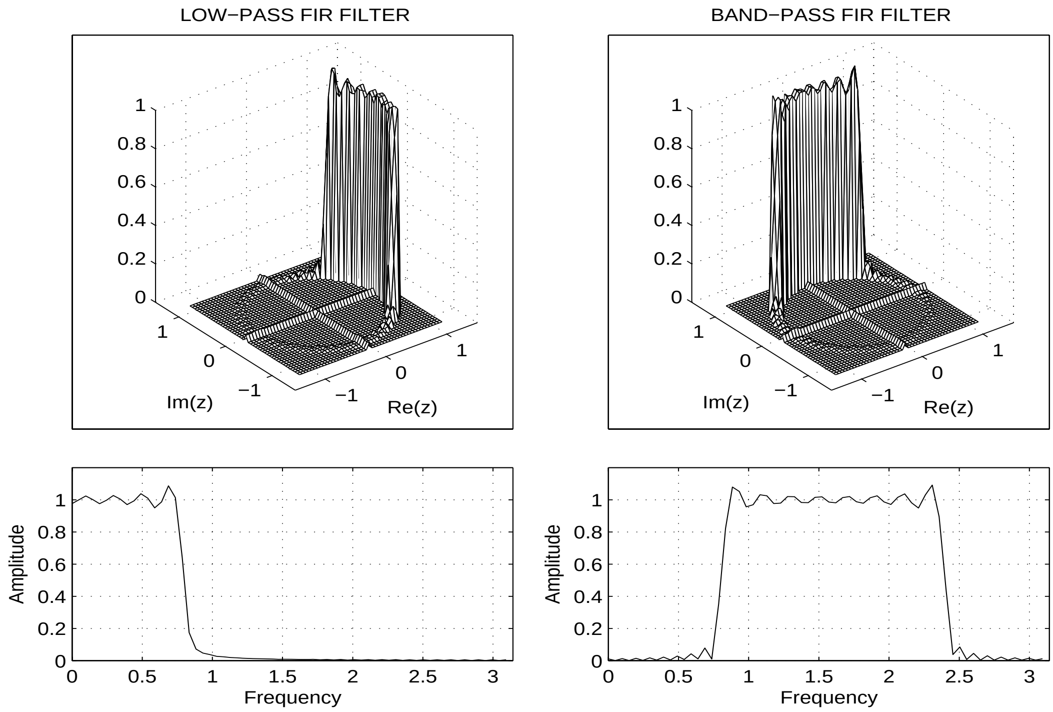

FIGURE 5.8. Magnitude frequency response of the low-pass and band-pass FIR filter

where

$$\tilde{h}(n) = h(n)e^{j\omega_s n} \tag{5.27}$$

represents the complex impulse response of the band-pass filter. The design procedure is summarised in Algorithm 5.1.

It is obvious that owing to symmetry properties of the discrete Fourier transform it is usually necessary to preserve this symmetry when designing digital filters.

Example 5.2 *Use the band-pass FIR filter of length* $P = 51$ *to extract the signal in the frequency band* $\langle 1, 2\rangle$ *[rad] from the original sequence*

$$x(n) = \sin(0.2n) + \sin(1.6n)$$

for $n = 0, 1, \cdots, N-1$.

Solution: Impulse response of the original low-pass filter for $\omega_c = 0.5$ [rad] is given by Eq. (5.23) which must be then modified by $\omega_s = 1.5$ according to Eq. (5.27) to define the complex impulse response

$$\tilde{h}(n) = \frac{\sin((n-(P-1)/2)\omega_c)}{\pi(n-(P-1)/2} e^{j\omega_s n}$$

for $n = 0, 1, \cdots, P-1$ standing for the band-pass filter in frequency band $\langle 1, 2\rangle$ [rad] presented in Fig. 5.9. Owing to symmetry properties it is further necessary to define the complex impulse response

$$\tilde{h}^*(n) = \frac{\sin((n-(P-1)/2)\omega_c)}{\pi(n-(P-1)/2} e^{-j\omega_s n}$$

of the complex conjugate filter. The given sequence processing can be based upon Eq. (5.15) in the form

$$y(n) = \sum_{k=0}^{P-1} (\tilde{h}(k)x(n-k) + \tilde{h}^*(k)x(n-k))$$

for $n = P, P+1, \cdots, N-1$.

Further impulse response modifications involve the application of other window described till now [23]. Their use mentioned in the previous section enable more efficient choice of the stop band attenuation and the transition width as well.

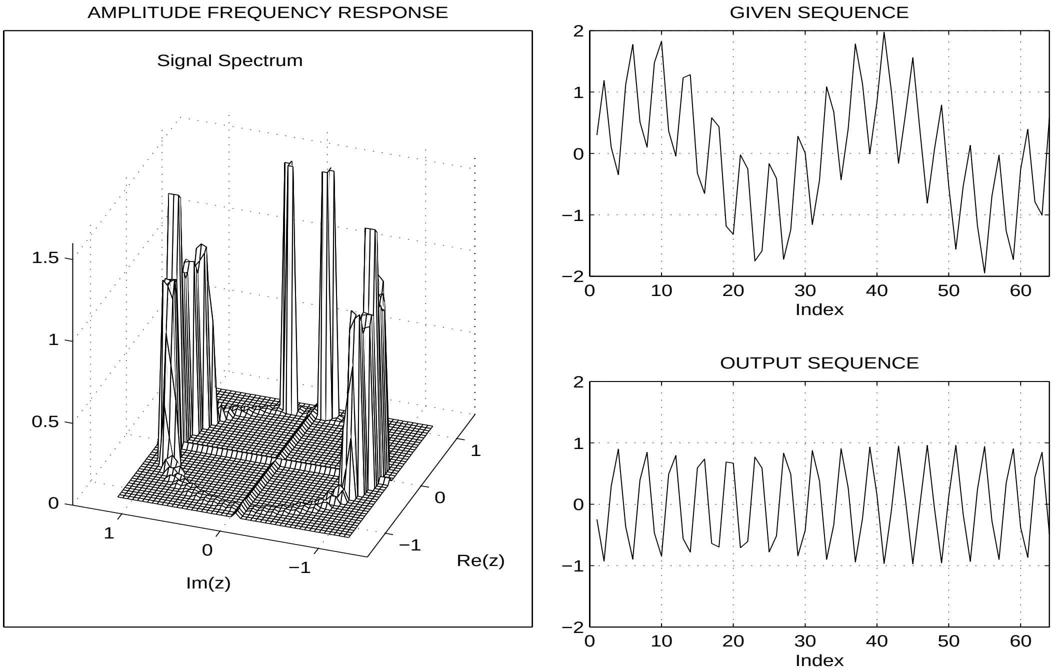

FIGURE 5.9. Band-pass processing of sequence $x(n) = \sin(0.2n) + \sin(1.6n)$

5.3 Multirate Digital Filters

The sampling rate in various parts of the digital system can be different which may be useful to reduce the number of computations and to decrease the number of values used for system analysis and information transition [2, p.234]. This principle is often applied for design of filter banks based on the DFT and assumes that the given sequence with sampling frequency f_s is used to form N_s sequences covering N_s separate frequency bands with their sampling rate reduced to f_s/N_s allowing their separate processing. This principle closely connected with the application of FIR filters is discussed further.

5.3.1 Band Limited Signal Sampling Rate Reduction

Let us restrict consideration of the previous section to the impulse response in the form

$$h(n) = \frac{\sin(n-(P-1)/2)\omega_c}{\pi(n-(P-1)/2)}$$

standing for the low-pass FIR filter with its cutoff frequency ω_c being a base for the band-pass filter having central frequency ω_s with its complex impulse response defined by

$$\tilde{h}(n) = h(n)e^{j\omega_s n} = \frac{sin(n-(P-1)/2)\omega_c}{\pi(n-(P-1)/2)} e^{j\omega_s n} \tag{5.28}$$

for $n = 0, 1, \cdots, P-1$. Let us choose $\omega_c = \pi/(2K)$ with K standing for number of separate frequency sub-bands of the normalized frequency range $\langle 0, \pi \rangle$. Defining further $\omega_s = \pi/(2K) + i\pi/K$ with principal channels defined for $0 \leq i \leq K-1$ we can use the complex impulse response (5.28) in the form

$$\tilde{h}_i(n) = \frac{\sin(n-(P-1)/2)\pi/(2K)}{\pi(n-(P-1)/2)} e^{j(\pi/(2K)+i\pi/K)n} = h(n)e^{j(\pi/(2K)+i\pi/K)n} \tag{5.29}$$

to realize the band-pass filter covering any frequency sub-band of the range $\langle 0, \pi \rangle$ and standing for one channel of a filter bank discussed further.

Example 5.3 *Evaluate the amplitude frequency response of a FIR band-pass filter of length $P = 64$, with $K = 4$ channels modified by the Hamming window and covering channels with indices 0 and 3*

Solution: Using the window function

$$W(n) = \alpha - (1-\alpha)\cos(2\pi n/(P-1))$$

for $n = 0, 1, \cdots, P-1$ and $\alpha = 0.54$ defined before to modify the complex impulse response given by Eq. (5.29) it is possible to apply the FFT to obtain results presented in Fig. 5.10. It is obvious that the passband having bandwidth $BW = \pi/K$ [rad/s] (equal to $1/(2K)$ [Hz]) is represented by $M = P/(2K)$ samples.

Assume a band limited real sequence $\{y(n)\}$ for $n = 0, 1, \cdots, N-1$ with its number of samples N as the integer multiple of P and its spectrum being inside one of the previously defined principal channel i for $i \in \langle 0, K-1\rangle$ and its complex conjugate. Its spectrum is defined by means of the discrete Fourier transform for $N = P$ in the form

$$Y(k) = \sum_{n=0}^{P-1} y(n) e^{-jkn2\pi/P} \tag{5.30}$$

for $k = 0, 1, \cdots, P-1$. Using the inverse discrete Fourier transform it is possible to evaluate

$$y(n) = \frac{1}{P}\sum_{k=0}^{P-1} Y(k) e^{jkn2\pi/P} \tag{5.31}$$

Example 5.4 *Evaluate the amplitude spectrum of sequence*

$$y(n) = \sin(2\pi 0.16 n)$$

for $n = 0, 1, \cdots, P-1$ and $P = 64$ and find band-pass filters dividing frequency range $\langle 0, \pi\rangle$ to $K = 4$ sub-bands and including this spectrum.

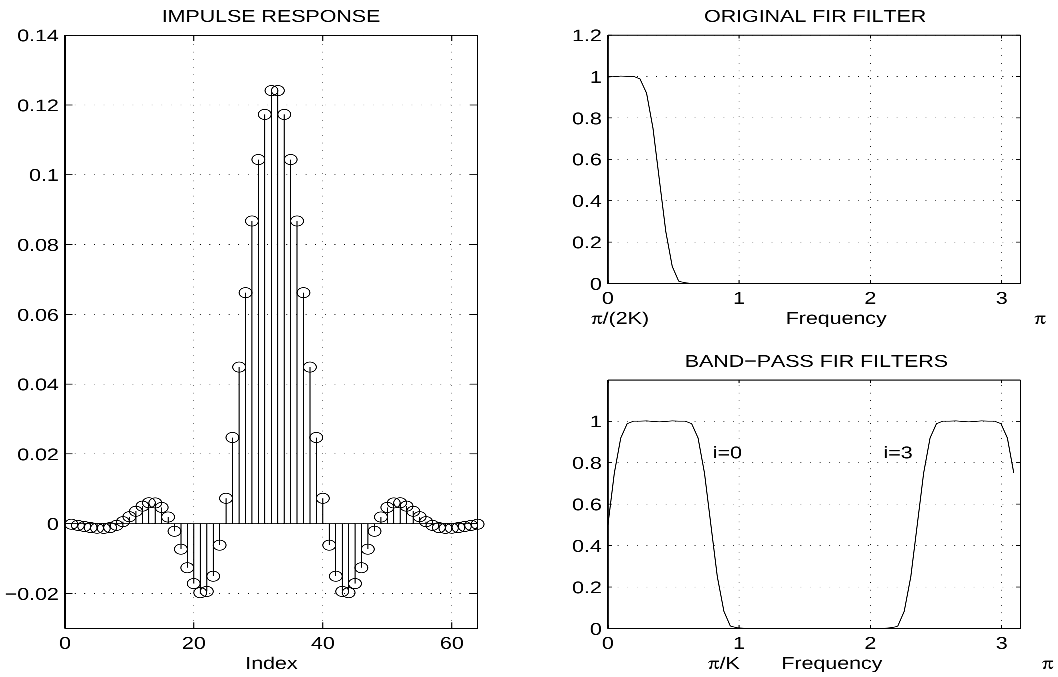

FIGURE 5.10. Basic impulse response of FIR low-pass filter of length $P = 64$ and bandwidth $BW = 1/(2K)$ for $K = 4$ channels modified by the Hamming window ($\alpha = 0.54$) with its frequency response and related band-pass filters spectra covering channels $i = 0$ and 3.

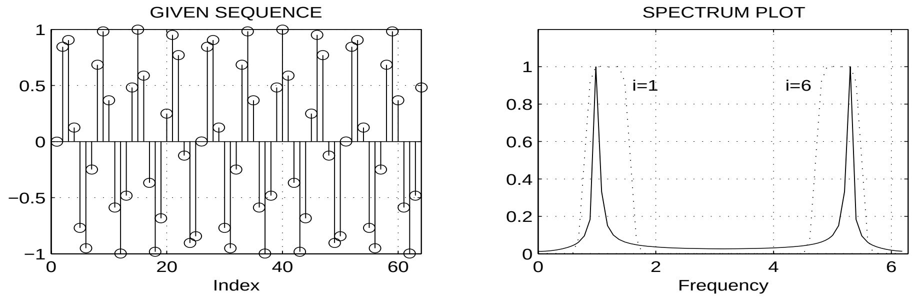

FIGURE 5.11. The given sequence $y(n) = \sin(2\pi 0.16n), n = 0, 1, \cdots, N-1$ for $N = 64$ and its spectrum with frequency responses of two band-pass filters of length $P = 64$ for $K = 4$, $i = 1$ and 6.

Solution: Spectrum of the given sequence has its frequency component $f = 0.16$ included in channel $i = 1$ covering the frequency band $\langle 0.125, 0.25\rangle$ and its complex conjugate with its index $i = 6$ according to Fig. 5.11.

THE SAMPLING RATE REDUCTION by the value K can be realized using each K-th value of the original sequence only and it is possible to show that this reduced sequence contains enough information enabling to reconstruct the whole original sequence again.

Theorem 5.2 *Assume that a given sequence* $\{y(n)\}, n = 0, 1, \cdots, P-1$ *has its spectrum estimate* $Y(k), k = 0, 1, \cdots, P-1$ *for normalized sampling frequency* $f_s = 1$ *in the range* $\langle 0, 1)$. *Then spectrum of the reduced sequence*

$$\{r(n) : r(n) = y(nK)\} \tag{5.32}$$

for $n = 0, 1, \cdots, 2M-1$ *where* $2M = P/K$ *stands for an even integer is related to the original one by equation*

$$R(m) = \frac{1}{K}\sum_{q=0}^{K-1} Y(m + q\,2M) \tag{5.33}$$

for $m = 0, 1, \cdots, 2M-1$ *covering the frequency range* $\langle 0, 1/K)$.

Proof: The Fourier transform of sequence $\{r(n)\}$ defined by (5.32) can be expressed in the form

$$R(m) = \sum_{n=0}^{2M-1} y(nK) e^{-jmn2\pi/(2M)} \tag{5.34}$$

for $m = 0, 1, \cdots, 2M-1$. After substitution for $y(nK)$ from Eq. (5.31) we obtain

$$R(m) = \sum_{n=0}^{2M-1} \frac{1}{P}\sum_{k=0}^{P-1} Y(k) e^{jknK2\pi/P} e^{-jmn2\pi/(2M)}$$

and taking into account that $P = 2MK$ it is possible to write

$$R(m) = \frac{1}{P}\sum_{k=0}^{P-1} Y(k) \sum_{n=0}^{2M-1} e^{jn(k-m)\pi/M}$$

As the second sum is nonzero for $(k-m) = q\,2M$ only where q is any integer and its value is equal to $2M = P/K$ it is possible to simplify the previous equation to the form given by Eq. (5.33). As stated before the frequency range for the original normalized sampling frequency $f_s = 1$ is equal to $\langle 0, f_s\rangle$. The subsampling reduces the sampling frequency to $f_r = f_s/K$ for sequence $\{r(n)\}$ and therefore its spectrum covers the frequency range $\langle 0, f_r\rangle$.

Result of this Theorem implies that the Fourier transform (5.30) of the original sequence having $P = 2MK$ values is reduced by subsampling defined by Eq. (5.32) to $2M$ values only which can be evaluated as the average of K original values according to Eq. (5.33).

Example 5.5 *Evaluate the spectrum of sequence $\{r(n)\}$ related to sequence*

$$y(n) = \sin(2\pi 0.16n)$$

studied in Example 5.4 by the reduction coefficient $K = 4$.

Solution: Results obtained from the definition of the DFT and relation (5.33) are presented in Fig. 5.12. The original sequence of length $P = 64$ is reduced to $P/K = 2M = 16$ values only. In the same way $P = 64$ spectrum values covering normalized frequency band $\langle 0, 1\rangle$ are reduced to $2M = 16$ values only defining frequency range $\langle 0, 1/K\rangle$. But owing to the band limited given sequence it is still possible to distinguish parts due to channel $i = 1$ and its complex conjugate even for the reduced sequence spectrum. This property is quite essential for further signal reconstruction.

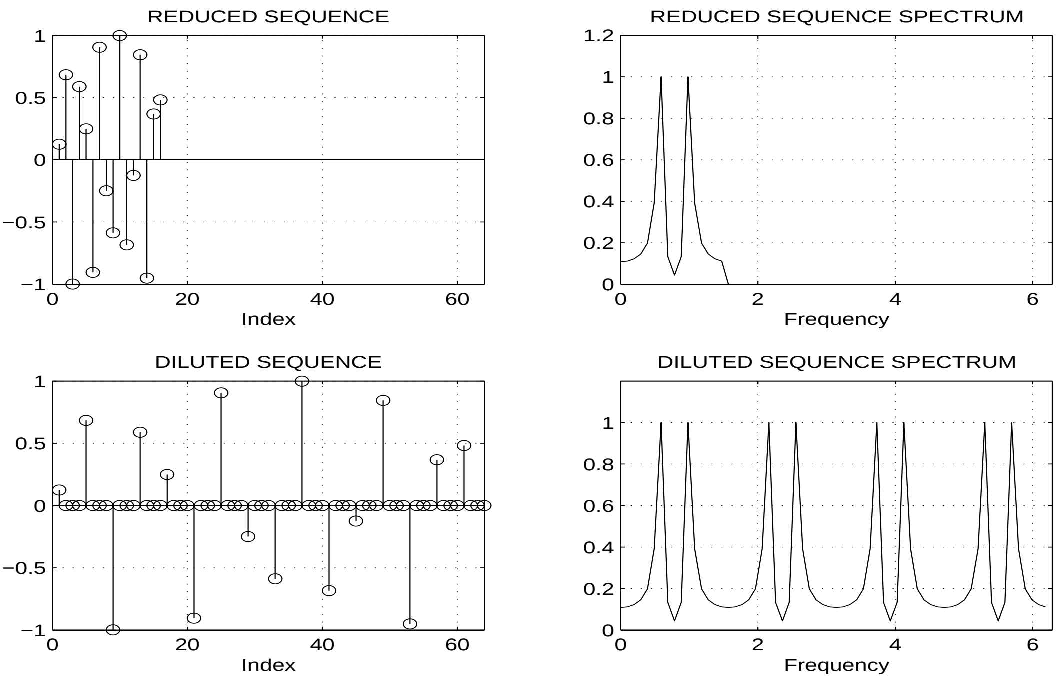

FIGURE 5.12. The reduced sequence formed by each K-th value of the original signal (for $K = 4$) and the diluted sequence with their spectra

Signal reconstruction based on the reduced sequence defined above can be used to verify that such a sequence can transmit enough information about the original band limited signal.

Theorem 5.3 *Let us dilute the reduced sequence $\{r(n): r(n) = y(nK)\}$ for $n = 0, \cdots, 2M-1$ where $2M = P/K$ stands for an even integer by $(K-1)$ zeros inserted among its each adjacent values to form sequence*

$$z(n) = \begin{cases} y(n) & \text{for } n = 0, K, 2K, \cdots, (2M-1)K \\ 0 & \text{for other values of } n \end{cases} \tag{5.35}$$

and let us assume that the spectrum of the original signal $\{y(n)\}$ is completely contained in channel i of the band-pass filter defined by its impulse response $\tilde{h}_i(n)$ given by Eq. (5.29) *and its complex conjugate (related to* $\tilde{h}_i^*(n)$*). Then the original signal can be reconstructed by sequence*

$$s(n) = \sum_{p=0}^{P-1} z(n-p)\ (\tilde{h}_i(n) + \tilde{h}_i^*(n)) \tag{5.36}$$

for $n = 0, 1, \cdots, P-1$*.*

Proof: The discrete Fourier transform of sequence given by Eq. (5.35) can be evaluated in the form

$$Z(k) = \sum_{n=0,K,2K,\cdots}^{(2M-1)K} y(n)e^{-jkn2\pi/P} = \sum_{q=0}^{2M-1} y(qK)e^{-jkqK2\pi/P}$$

for $k = 0, 1, \cdots, P-1$. Taking into account that $P = 2MK$ we get sequence

$$Z(k) = \sum_{q=0}^{2M-1} y(qK)e^{-jkq2\pi/(2M)} \tag{5.37}$$

representing the periodic sequence with its period $2M$ samples long and owing to Eq. (5.34) standing for the periodic extension of the DFT of the reduced sequence $\{r(n)\}$. This fact implies that the original spectrum of sequence $\{y(n)\}$ can be evaluated by the scalar multiplication of spectrum $Z(k)$ by the frequency response of channel i and its complex conjugate. According to properties of DFT this multiplication corresponds to the convolution in time domain given by Eq. (5.36).

Example 5.6 *Evaluate the spectrum of sequence* $\{z(n)\}$ *related to sequence*

$$y(n) = \sin(2\pi 0.16n)$$

by Eq. (5.35) and use the convolution with the impulse response $\tilde{h}_i(n)$ *and its complex conjugate for channel* $i = 1$ *to estimate the original sequence.*

Solution: Results obtained from the definition of the DFT or relation (5.37) are presented in Fig. 5.12. It is possible to see that for $P = 64$ samples of sequence $\{z(n)\}$ its spectrum covering the frequency range $\langle 0, 1\rangle$ [Hz] by $P = 64$ values is periodic with its period $P/K = 2M = 16$ samples long which is equal to frequency band $1/K$. Using Eq. (5.36) it is possible to reconstruct the original sequence in the form presented in Fig. 5.13.

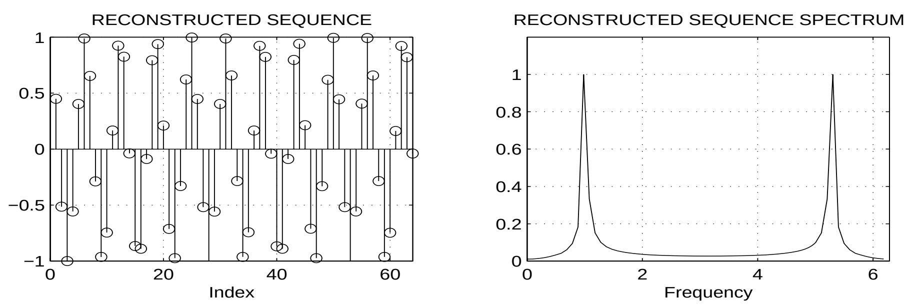

FIGURE 5.13. The output sequence $\{s(n)\}$ defined by convolution of the diluted sequence and two complex conjugate impulse responses defining frequency bands of the original signal and the output sequence spectrum

It is possible to summarize results of the previous mathematical description in the following way

- The real signal sampled with the frequency f_s (according to sampling theorem) being band limited to the couple of complex conjugate frequency bands each $f_s/(2K)$ long may be subsampled in such way that each K-th sample is used for further processing only

- The narrover the frequency band is the more substantial subsampling may be applied allowing lower number of computations involved.

Results given above are very important for design of multirate adaptive filters [2] and filter banks mention further.

5.3.2 Filter Bank Design

A filter bank [22] is very important signal processing tool usually based on band-pass FIR filters and often using the multirate principle as well.

Let us assume that we would like to design a filter bank covering frequency range $\langle 0, \pi \rangle$ by K band-pass filters. Each of these channels can be defined by its complex finite impulse response given by Eq. (5.29) enabling to find its output for a given sequence $\{x(n)\}$ in the form

$$y_i(n) = \sum_{p=0}^{P-1} \tilde{h}_i(p) x(n-p) \tag{5.38}$$

Output signal can then be evaluated by equation

$$y(n) = \sum_{i=0}^{2K-1} v_i y_i(n) \tag{5.39}$$

where coefficients v_i enable to define gain of separate channels and in case of their equal value $v_i = 1$ for $i = 1, 2, \cdots, 2K-1$. This structure can be used in case of the adaptive signal processing as well with individual gains changed during the learning process of the system.

Example 5.7 *Evaluate the frequency response of a filter bank of length $P = 64$ for $K = 4$ channels.*

Solution: Using an impulse as the input sequence $\{x(n)\}$ in Eq. (5.38) it is possible to evaluate the output sequence $\{y(n)\}$ by Eq. (5.39) for $v_i = 1$, $i = 1, 2, \cdots, 2K-1$ and its spectrum. Fig. 5.14 presents resulting frequency response for application of rectangular and Hamming window as well.

Problems connected with the realization of the computational algorithm involve

- design of such a filter bank enabling perfect reconstruction of the original signal
- application of the FFT in process of signal decomposition and reconstruction given by Eq. (5.38) and (5.39) to realize faster processing
- proper subsampling in a multirate filter bank enabling to reduce the number of computations

Possibilities of subsampling and reduction of computations are very important especially for the real time applications as the time consumption for long sequences processing may be substantially decreased.

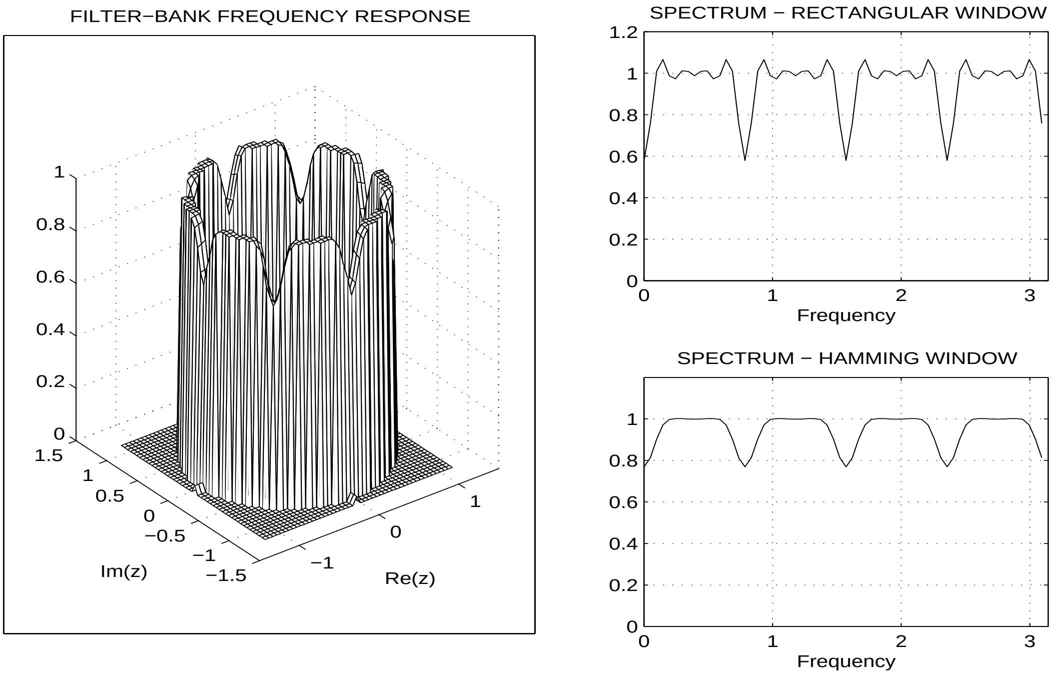

FIGURE 5.14. Normalized frequency response of $K = 4$ channel FIR filter bank of length $P = 64$ samples with the use of rectangular and Hamming window function and its sketch in the complex plane.

Example 5.8 *Find the fast algorithm evaluating the outputs of all* $2K$ *filter bank channels using the FIR band-pass filters of length* P.

Solution: Assuming that $P = 2MK$ and denoting $p = q + r2K$ we can write Eq. (5.38) in the form

$$y_i(n) = \sum_{q=0}^{2K-1} \sum_{r=0}^{M-1} \tilde{h}_i(q + r2K)x(n - q - r2K)$$

Using Eq. (5.29) and owing to periodicity

$$\begin{aligned} \tilde{h}_i(q + r2K) &= h(q + r2K)e^{j(\pi/(2K)+i\pi/K)(q+r2K)} = \\ &= h(q + r2K)e^{j\pi(q+r2K)/(2K)}e^{jiq2\pi/(2K)} \end{aligned}$$

and therefore

$$\begin{aligned} y_i(n) &= \sum_{q=0}^{2K-1} e^{jiq2\pi/(2K)} \sum_{r=0}^{M-1} h(q + r2K)x(n - q - r2K)e^{j\pi(q+r2K)/(2K)} = \\ &= \sum_{q=0}^{2K-1} g(n, q)e^{jiq2\pi/(2K)} \end{aligned} \tag{5.40}$$

The last expression implies that it is possible for each n to evaluate $g(n, q)$ at first and then use the inverse DFT for evaluation of $y_i(n)$ for all indices i at once. In case that $2K$ is a power of 2 the inverse FFT can be applied.

A simple method of the filter bank application based on Eq. (5.40) and (5.39) is presented in Algorithm 5.2. Further improvement of this algorithm may be achieved by the use of multirate processing.

Algorithm 5.2 *Processing of a sequence* $\{x(n)\}_{n=1}^{2P}$ *by a filter bank composed of* K *FIR filters of length* P.

- definition of the filter length P, number of channels K and masking vector $\mathbf{v} = [v(1), v(2), \cdots, v(2K)]$
- definition of vector $\mathbf{x} = [x(1), x(2), \cdots, x(2P)]$
- definition of the basic impulse response

$$\mathbf{h} = \sin((0:P-1) - (P-1)/2 * \pi/(2*K))/(\pi((0:P-1) - (P-1)/2))$$

- for $n = P, P+1, \cdots, 2P$
  - evaluation of
    $$g(q) = x((n-q) : -2K : (n-q-(M-1)*2K))* (h(q : 2K : (q+(M-1)*2K)).* \exp(j\pi(q+(1:M-1)*2K)/(2*K)))';$$
    for $q = 0, 1, \cdots, 2K-1$ and $M = P/(2K)$ used in Eq. (5.40)
  - evaluation of outputs $\mathbf{f} = [f(1), \cdots, f(2K)]$ of separate bank channels based on Eq. (5.40)
    $\mathbf{f} = \text{fft}(\mathbf{g})$;
  - evaluation of system output according to (5.39)
    $y(n) = \text{sum}(\mathbf{f}. * \mathbf{v})$;

The multirate processing can be very efficient as it allows not to follow the sampling theorem on a channel-by-channel basis as it is sufficient to meet it on the sum of channels. The evaluation of outputs of K separate channels given by Eq. (5.38) or (5.40) can then be performed for $n = 0, K, 2K, \cdots$ only. The diluted sequences $\{z_i(n)\}$ defined by equation

$$z_i(n) = \begin{cases} y_i(n) & \text{for } n = 0, K, 2K, \cdots \\ 0 & \text{for other values of } n \end{cases} \tag{5.41}$$

similar to Eq. (5.35) can then be used for signal reconstruction. Modifying Eq. (5.36) for complex signals restricted to one channel only we can use Eq. (5.39) in the form

$$y(n) = \sum_{i=0}^{2K-1} B_i \sum_{p=0}^{P-1} z_i(n-p)\tilde{h}_i(p) \tag{5.42}$$

After a few modifications [34] the FFT can be used in this stage as well.

It is obvious that in case of a complex band limited signal to a single channel even stronger subsampling may be applied with such a reduction that only each (2K)-th sample is used.

Example 5.9 *Use the filter bank composed of* $K = 8$ *FIR filters of length* $P = 128$ *to extract signal components in its channels* 1 *and* 5 *for*

$$x(n) = \sin(2\pi f_1 n) + \sin(2\pi f_2 n) + 0.2\sin(2\pi f_3 n)$$

for $f_1 = 0.1,\ f_2 = 0.15,\ f_3 = 0.35$ *and* $n = 0, 1, \cdots, 2P-1$.

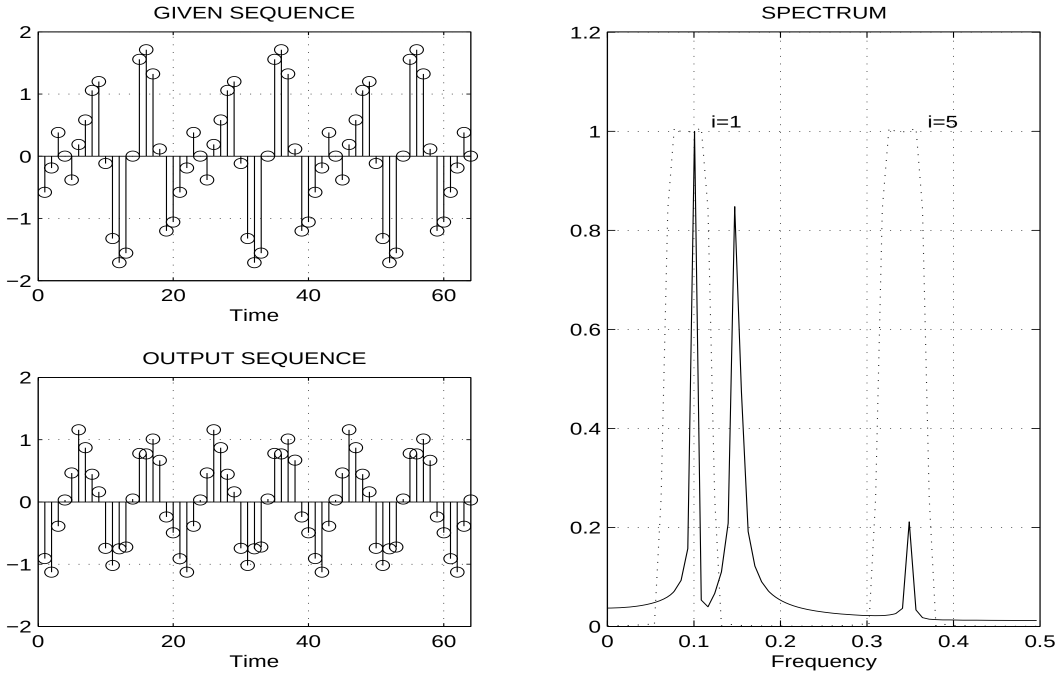

FIGURE 5.15. Application of a filter bank of length $P = 128$ and $K = 8$ channels to process the given sequence with active channel 1 and 5

Solution: Using the Algorithm 5.2 it is possible to evaluate the system output in form presented in Fig. 5.15 together with the signal spectrum and filter bank frequency response. It is possible to see that the middle frequency component is excluded.

The last example presented the non-adaptive application of a filter bank with gains of separate channels defined in advance. In many other applications their gain can be adapted with respect to changing conditions in the observed system.

5.4 Infinite Impulse Response Filters

Digital filter is simply specified by the *discrete transfer function* $H(z)$ with its behaviour defined by the *frequency response* $H(e^{j\omega_i}) = H(z) \mid_{z=e^{j\omega_i}}$. We shall study now a general transfer function given as a rational function implying the sequence $\{h(n)\}$ standing for the *infinite impulse response* in the form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{N} a(k) z^{-k}} = \sum_{k=0}^{\infty} h(k) z^{-k} \tag{5.43}$$

In comparison with the FIR system not so many coefficients are necessary in this case enabling to process the given sequence by the *difference equation*

$$y(n) = -\sum_{k=1}^{N} a(k) y(n-k) + \sum_{k=0}^{N} b(k) x(n-k) \tag{5.44}$$

On the other hand the problem of stability must be followed for IIR systems and the phase characteristic is not linear as in FIR processes.

There are various digital filter design procedures enabling to evaluate coefficients of the transfer function or the difference equation [30, 32, 23] and we shall mention some of them only.

5.4.1 Least Squares Frequency Characteristics Approximation

Let us suppose a given set of the magnitude frequency response values $|H_d(e^{j\omega_i})|$ for frequencies ω_i, $i = 0, 1, \cdots, M-1$. Using the discrete transfer function given by Eq. (5.43) we can apply the least square method to evaluate its vector of coefficients

$$\mathbf{w} = [b(0), b(1), \cdots, b(N), a(1), a(2), \cdots, a(N)] \tag{5.45}$$

minimizing the sum of squared errors defined as

$$J(\mathbf{w}) = \sum_{i=0}^{M-1} \left(\left|H(e^{j\omega_i})\right| - \left|H_d(e^{j\omega_i})\right| \right)^2 \tag{5.46}$$

under constraints that all poles of the discrete transfer function are inside the unit circle.

Theorem 5.4 *Let us assume the discrete transfer function $H(z)$. Then*

$$\left|H(e^{j\omega_i})\right|^2 = H(z)\ H(z^{-1})\ |_{z=e^{j\omega_i}} \tag{5.47}$$

Proof: As $H(e^{j\omega_i}) = H(z)\ |_{z=e^{j\omega_i}}$ has a complex value for each ω_i its magnitude results from the product of complex conjugate values of $H(e^{j\omega_i})$ and $H(e^{-j\omega_i})$.

Using Eqs. (5.43) and (5.47) in Eq. (5.46) we obtain the sum of squared errors in the form

$$J(\mathbf{w}) = \sum_{i=0}^{M-1} \left(\sqrt{\frac{\sum_{k=0}^{N} b(k) e^{-jk\omega_i} \sum_{k=0}^{N} b(k) e^{jk\omega_i}}{(1 + \sum_{k=1}^{N} a(k) e^{-jk\omega_i})(1 + \sum_{k=1}^{N} a(k) e^{jk\omega_i})}} - \left|H_d(e^{j\omega_i})\right| \right)^2 \tag{5.48}$$

After the evaluation of the gradient $\mathbf{g} = \partial J(\mathbf{w})/\partial \mathbf{w}$ we can use the steepest descent method mentioned above to minimize the squared error given by Eq. (5.48). As no control over stability is applied in this basic method the resulting poles outside the unit circle must be modified using the following statement.

Theorem 5.5 *The amplitude frequency response of system having its discrete transfer function with the pole component $(z – Ae^{j\varphi})$ is the same as that with component $A(z – \frac{1}{A} e^{j\varphi})$.*

Proof: Studying magnitudes of the frequency response only it is possible to evaluate

$$\begin{aligned} M1 &= \left|e^{j\omega} - Ae^{j\varphi}\right| = \sqrt{(\cos\omega - A\cos\varphi)^2 + (\sin\omega - A\sin\varphi)^2} = \\ &= \sqrt{1 + A^2 - 2A\cos(\omega - \varphi)} \end{aligned}$$

Similarly

$$\begin{aligned} M2 &= \left|e^{j\omega} - \frac{1}{A} e^{j\varphi}\right| = \sqrt{(\cos\omega - \frac{1}{A}\cos\varphi)^2 + (\sin\omega - \frac{1}{A}\sin\varphi)^2} = \\ &= \sqrt{1 + \frac{1}{A^2} - \frac{2}{A}\cos(\omega - \varphi)} = \frac{1}{A}.M1 \end{aligned}$$

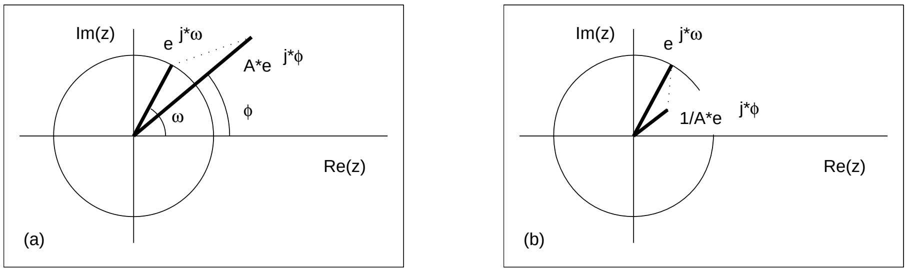

FIGURE 5.16. Illustration of poles resulting in the same amplitude frequency responses

Both cases are presented in Fig. 5.16.

This statement allows to use reciprocal poles instead of the original ones laying outside the unit circle which after the gain correction preserve the magnitude response and result in the stable discrete system.

Example 5.10 *Evaluate coefficients of the discrete transfer function (5.43) for $N = 3$ related to the difference Eq. (5.44) approximating the ideal low-pass filter with its magnitude frequency response*

$$H_d(e^{j\omega_i}) = \begin{cases} 1 & for\ i = 0, 1, 2, 3 \\ 0 & for\ i = 4, 5, \cdots, M-1 \end{cases}$$

where $\omega_i = i\frac{\pi}{M}$, $M = 10$.

Solution: Using the following notation

$$g(\omega_i) = \sum_{k=0}^{N} b(k)e^{-jk\omega_i}, \quad g^*(\omega_i) = \sum_{k=0}^{N} b(k)e^{jk\omega_i}$$

$$f(\omega_i) = 1 + \sum_{k=1}^{N} a(k)e^{-jk\omega_i}, \quad f^*(\omega_i) = 1 + \sum_{k=1}^{N} a(k)e^{jk\omega_i}$$

it is possible to express Eq. (5.48) in the form

$$J(\mathbf{w}) = \sum_{i=1}^{M-1} (\sqrt{\frac{g(\omega_i)g^*(\omega_i)}{f(\omega_i)f^*(\omega_i)}} - |H_d(e^{j\omega_i})|)^2$$

with an asteric standing for the complex conjugate. Using further notation given by (5.45) it is possible to evaluate gradients for $k = 0, 1, \cdots, N$ in the form

$$\frac{\partial J(\mathbf{w})}{\partial b_k} = 2\sum_{i=1}^{M-1} \left(\sqrt{\frac{g(\omega_i)g^*(\omega_i)}{f(\omega_i)f^*(\omega_i)}} - \left|H_d(e^{j\omega_i})\right| \right) \cdot \frac{1}{2}\sqrt{\frac{f(\omega_i)f^*(\omega_i)}{g(\omega_i)g^*(\omega_i)}} \cdot \frac{e^{-jk\omega_i}g^*(\omega_i) + g(\omega_i)e^{jk\omega_i}}{f(\omega_i)f^*(\omega_i)}$$

and for $k = 1, 2, \cdots, N$ by the following expression

$$\frac{\partial J(\mathbf{w})}{\partial a_k} = 2\sum_{i=1}^{M-1} \left(\sqrt{\frac{g(\omega_i)g^*(\omega_i)}{f(\omega_i)f^*(\omega_i)}} - \left|H_d(e^{j\omega_i})\right| \right) \cdot \frac{1}{2}\sqrt{\frac{f(\omega_i)f^*(\omega_i)}{g(\omega_i)g^*(\omega_i)}}$$
$$\cdot \frac{-g(\omega_i)g^*(\omega_i)[e^{-jk\omega_i}f^*(\omega_i) + f(\omega_i)e^{jk\omega_i}]}{(f(\omega_i)f^*(\omega_i))^2}$$

Starting with the initial choice of the parameter vector $\mathbf{w} = [0.1, 0.2, 0.2, 0.1, -0.7, 0.5, -0.1]$ it is possible to use the method of the steepest descent for its updating in the form

$$\mathbf{w}_{new} = \mathbf{w}_{old} - c\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} \quad (5.49)$$

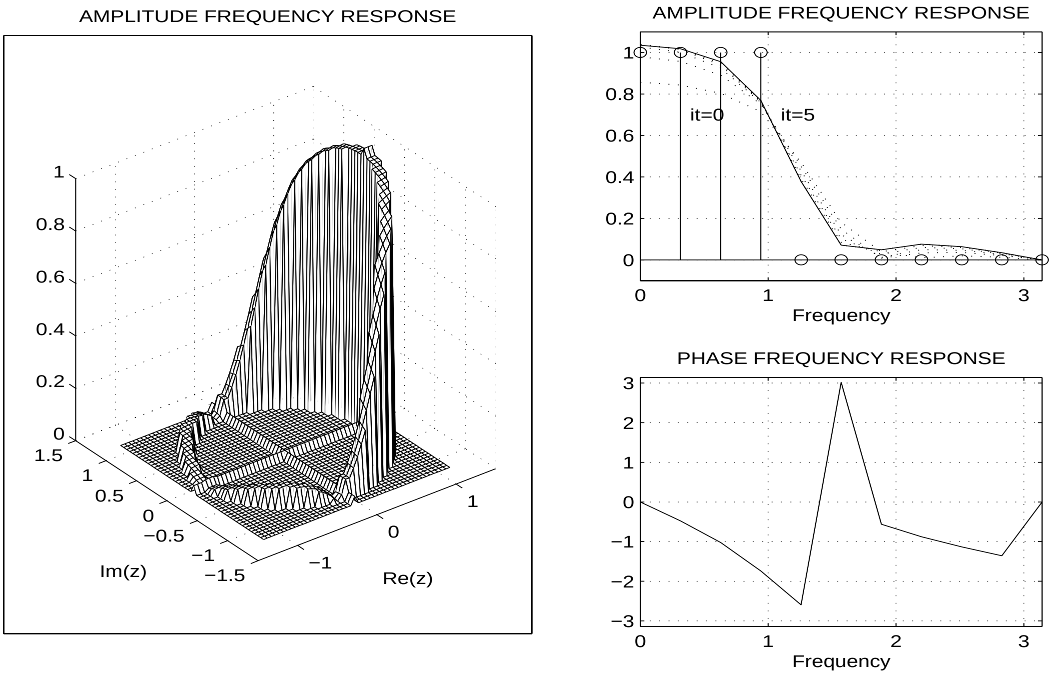

FIGURE 5.17. Result of the approximation of the desired frequency response with the process of the amplitude frequency response approximation

Results for the convergence factor $c = 0.01$ after 5 iterations and system stabilization is presented in Fig. 5.17.

Computer processing of the discrete transfer function design for a given magnitude frequency response is presented in Algorithm 5.3 using the MATLAB notation. The whole procedure with further modifications is included in the special function enabling the coefficients estimation in the form

$$[\mathbf{b}, \mathbf{a}] = \text{yulewalk}(N, \mathbf{u}, \mathbf{d})$$

for desired frequency response $\mathbf{d}$ defined for frequencies $\mathbf{u}$.

Example 5.11 *Use coefficients evaluated in Example 5.10 to process sequence*

$$x(n) = \sin(0.3n) + \sin(2.2n) \tag{5.50}$$

Solution: Using difference Eq. (5.44) for $N = 3$ in the form

$$\begin{aligned} y(n) = & - a(1)y(n-1) - a(2)y(n-2) - a(3)y(n-3) + \\ & + b(0)x(n) + b(1)x(n-1) + b(2)x(n-2) + b(3)x(n-3) \end{aligned}$$

it is possible to find solution presented in Fig. 5.18.

5.4.2 Analog System Simulation

As direct digital IIR filter design assumes rather complicated mathematical procedure analog systems are often used to be transformed into their discrete form. This approach is described in many books [30, 23, 5] and is based upon the chosen frequency characteristics in the analytical form and the corresponding transfer function evaluation.

We shall briefly mention the Butterworth filter only with its monotonous magnitude frequency response and squared gain factor for its low-pass version and normalized unit sampling in the form

$$\left|H(e^{j\omega})\right|^2 = \frac{1}{1+\frac{\tan^{2N}(\omega/2)}{\tan^{2N}(\omega_c/2)}} \tag{5.52}$$

for a given cutoff frequency ω_c. This characteristics presented in Fig. 5.19 for various filter orders N and chosen $\omega_c = 1$ approaches an ideal low-pass filter as N gets larger.

Relation

$$\left|H(e^{j\omega})\right|^2 = H(z)\ H(z^{-1})\ |_{z=e^{j\omega}}$$

implies that

$$H(z).H(z^{-1}) = \left|H(e^{j\omega})\right|^2_{e^{j\omega}=z} = \frac{\tan^{2N}(\omega_c/2)}{\tan^{2N}(\omega_c/2)+\tan^{2N}(\omega/2)}\ |_{e^{j\omega}=z}$$

As

$$\tan(\omega/2) = \frac{\sin(\omega/2)}{\cos(\omega/2)} = \frac{\frac{1}{j}(e^{j\omega/2}-e^{-j\omega/2})}{e^{j\omega/2}+e^{-j\omega/2}} = \frac{1}{j}\frac{e^{j\omega}-1}{e^{j\omega}+1}$$

the previous equation can be rewritten into the form

$$\begin{aligned}|H(z)|^2 &= \frac{\tan^{2N}(\omega_c/2)}{\tan^{2N}(\omega_c/2)+(-1)^N\left(\frac{z-1}{z+1}\right)^{2N}} \\ &= \tan^{2N}(\omega_c/2)\frac{(z+1)^{2N}}{(z+1)^{2N}\tan^{2N}(\omega_c/2)+(-1)^N(z-1)^{2N}}\end{aligned}$$

Evaluating $2N$ poles of this rational function it is possible to find [5] that N of them is inside the unit circle defining the discrete transfer function in the form

$$H(z) = c\,\frac{(z+1)^N}{(z-p(1))(z-p(2))\cdots(z-p(N))} \tag{5.53}$$

Coefficient c can be evaluated to achieve $|H(e^{j\omega})| = 1$ for $\omega = 0$ resulting in equation

$$c = \frac{1}{2^N}(1-p(1))(1-p(2))\cdots(1-p(N)) \tag{5.54}$$

Evaluating values of the characteristic polynomial it is further possible to define the discrete transfer function in the form

$$H(z) = c\,\frac{1+\begin{pmatrix}N\\1\end{pmatrix}z^{-1}+\begin{pmatrix}N\\2\end{pmatrix}z^{-2}+\cdots+\begin{pmatrix}N\\N\end{pmatrix}z^{-N}}{1+a(1)z^{-1}+\cdots+a(N)z^{-N}} \tag{5.55}$$

The whole design procedure is presented in Algorithm 5.4.

Example 5.12 *Evaluate coefficients of the low-pass Butterworth filter of order $N = 4$ having cutoff frequency $\omega_c = 1$ and use it to process sequence*

$$x(n) = \sin(0.3n) + \sin(2.2n)$$

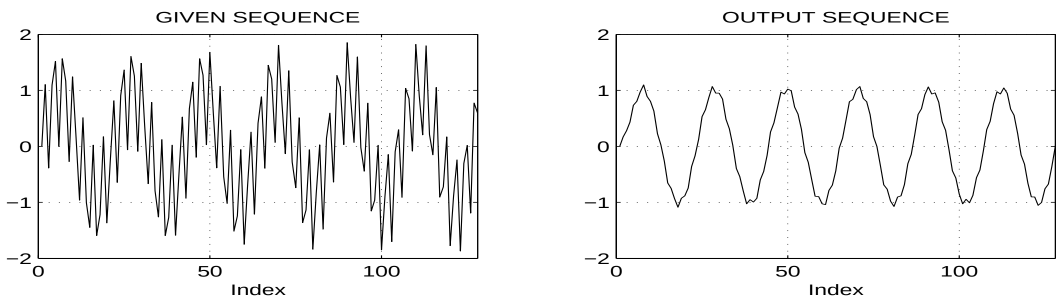

FIGURE 5.18. Sequence $x(n)=\sin(0.3n)+\sin(2.2n)$ before and after processing by an IIR filter

Algorithm 5.3 *Infinite impulse response filter design for*

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{N} a(k) z^{-k}} \tag{5.51}$$

and desired magnitude frequency response

$$\mathbf{d} = [d(1), d(2), \cdots, d(M)]$$

for frequencies in range $< 0, \pi >$ normalized to range $\langle 0, 1 \rangle$ defined by vector

$$\mathbf{u} = [u(1), u(2), \cdots, u(M)]$$

- choice of initial parameters $\mathbf{w} = [\mathbf{b}, \mathbf{a}]$ for $\mathbf{b} = [b(0), \cdots, b(N)]$, $\mathbf{a} = [a(1), \cdots, a(N)]$ and estimation of convergence factor c
- definition of frequency matrix

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp(-ju_1) & \exp(-ju_2) & \cdots & \exp(-ju_M) \\ & & \cdots & \\ \exp(-jNu_1) & \exp(-jNu_2) & \cdots & \exp(-jNu_M) \end{bmatrix}$$

- iterative process including
  - evaluation of vector $\mathbf{g} = \mathbf{b} * \mathbf{E}$ and $\mathbf{g}^*$ =conj($\mathbf{g}$)
  - evaluation of vector $\mathbf{f} = [\mathbf{1}, \mathbf{a}] * \mathbf{E}$ and $\mathbf{f}^*$ =conj($\mathbf{f}$)
  - amplitude frequency response evaluation
    $\mathbf{h}$ =sqrt($\mathbf{g} * \mathbf{g}^*$)./($\mathbf{f} * \mathbf{f}^*$))
  - gradient evuluation for $k = 0, 1, \cdots, N-1$
    $\mathbf{q}(k{+}1)$=sum(($\mathbf{h}-\mathbf{d}$).$*\mathbf{h}$.$\wedge(-1)$.$*$($\mathbf{E}(k,:)$.$*\mathbf{g}^*$+$\mathbf{g}$.$*$conj($\mathbf{E}(k,:)$)))./($\mathbf{f} * \mathbf{f}^*$)
  - gradient evaluation for $k = 1, 2, \cdots, N$
    $\mathbf{q}(k{+}N{+}1) = -$sum(($\mathbf{h}-\mathbf{d}$). $*$ $\mathbf{h}$. $\wedge$ (-1). $*$ ($\mathbf{g}$. $*$ $\mathbf{g}^*$)./($\mathbf{f}$. $*$ $\mathbf{f}^*$)...
    . $*$ ($\mathbf{E}(k,:) * \mathbf{f}^* + \mathbf{f}*$conj($\mathbf{E}(k,:)$)))
  - coefficients updating
    $\mathbf{w} = \mathbf{w} - c\mathbf{q}$; $\mathbf{b} = \mathbf{w}(1 : N+1)$; $\mathbf{a} = \mathbf{w}(N+2 : 2*N+1)$;
- final roots evaluation
  $\mathbf{r}$ =roots($\mathbf{a}$);
  with possible modification for $k = 1 : N$
  if abs($r(k)$) > 0
  $\mathbf{b} = \mathbf{b}$./abs($r(k)$);
  $r(k) = 1$/abs($r(k)*$exp($j*$angle($r(k)$)));
  end
  and new coefficients definition
  $\mathbf{a}$ =poly($\mathbf{r}$);
- final frequency plot presentation
  [$\mathbf{h}, \mathbf{u}$] =freqz($\mathbf{b}, \mathbf{a}, M$); plot($\mathbf{u}$,abs($\mathbf{h}$);

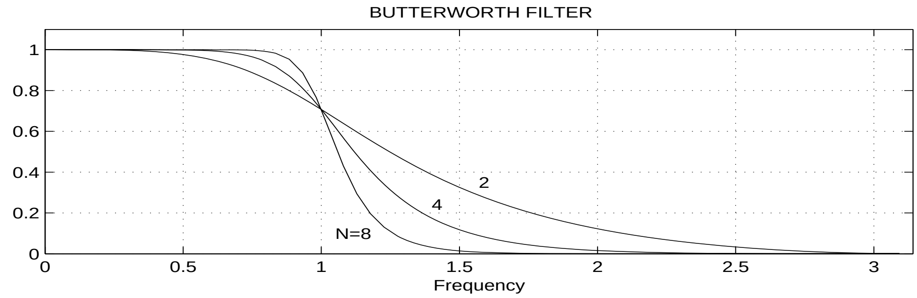

FIGURE 5.19. Magnitude frequency response of the low-pass Butterworth filter with its cutoff frequency $\omega_c = 1$.

Solution: Resulting difference equation

$$y(i) = -a(1)y(i-1) - \cdots - a(4)y(i-4) + c(x(i) + 4x(i-1) + 6x(i-2) + 4x(i-3) + x(i-4)) \tag{5.56}$$

provides results presented in Fig. 5.20.

Simple modification of the method described above enables realization of the high-pass or band-pass filter [5] as well.

5.5 Frequency Domain Signal Processing

Signal processing in frequency domain based upon the *discrete Fourier transform* is often used for direct modifications of the signal spectrum. This approach assumes no evaluation of the difference equation coefficients as the filtering process is realized in the frequency domain only [18, p.292, 369]. Separate steps of the evaluation procedure consist of

- the transformation of the given signal $\{x(n)\}_{n=0}^{N-1}$ into the frequency domain by the DFT

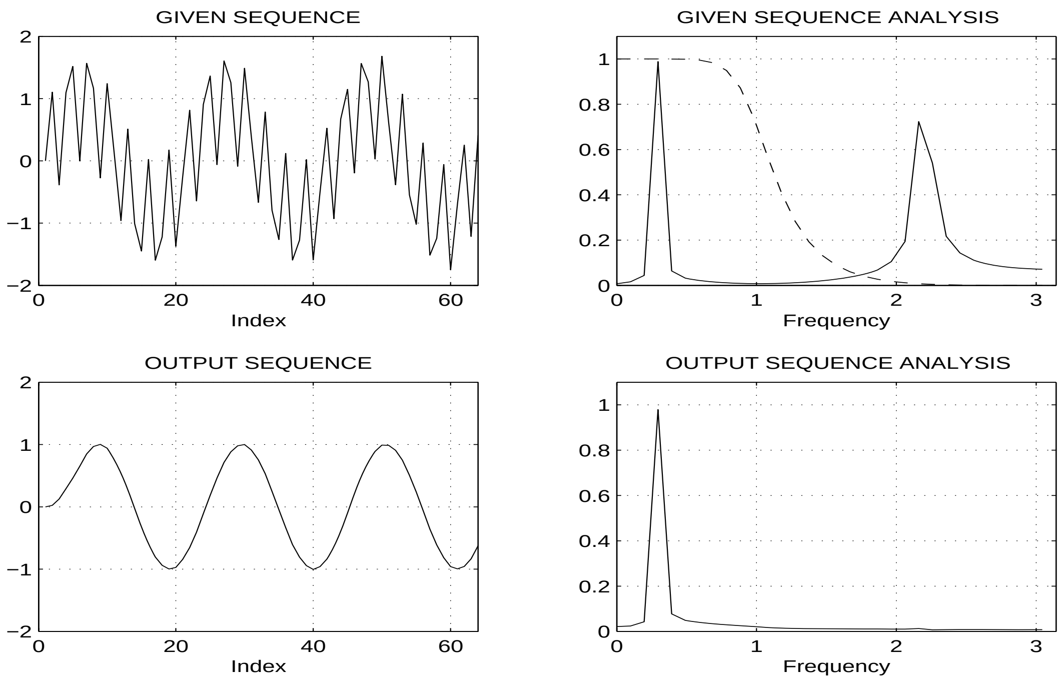

FIGURE 5.20. Results of the given sequence processing by the low-pass Butterworth filter of order $N = 4$

Algorithm 5.4 *The Butterworth low-pass filter design with its discrete transfer function*

$$H(z) = c\frac{(z+1)^N}{(z-p(1))(z-p(2))\cdots(z-p(N))} \tag{5.57}$$

- the choice of order N and cutoff frequency ω_c
- selection of N poles laying inside the unit circle from poles

$$p(i) = \frac{1-\tan^2(\omega_c/2) + j2\tan(\omega_c/2)\sin(\theta(i))}{1-2\tan(\omega_c/2)\cos(\theta(i)) + \tan^2(\omega_c/2)}$$

  for $i = 1, 2, \cdots, 2N$ where

$$\theta(i) = \begin{cases} (i-1)\pi/N & \text{for } n \text{ old} \\ (2i-1)\pi/(2N) & \text{for } n \text{ even} \end{cases}$$

- definition of coefficient c
  $c = \text{prod}(1-\mathbf{p})/2^N$
- characteristic polynomial evaluation
  $\mathbf{a} = \text{poly}(\mathbf{p})$
  defining the difference equation

$$\begin{aligned} y(i) = & - a(1)y(i-1) - a(2)y(i-2) - \cdots - a(N)y(i-N) \\ & + c(x(i) + \binom{N}{1}x(i-1) + \binom{N}{2}x(i-2) + \cdots + \binom{N}{N}x(i-N)) \end{aligned}$$

- multiplication of the frequency samples $\{X(k)\}_{k=0}^{N-1}$ by values of the desired frequency window function $\{H(k)\}_{k=0}^{N-1}$ to obtain modified frequency response $\{Y(k)\}_{k=0}^{N-1}$
- the transformation to the time domain through the inverse DFT to obtain signal $\{y(n)\}_{n=0}^{N-1}$

The FFT transform can be applied for this process as well and all necessary steps are summarised using MATLAB notation in Algorithm 5.5 with presentation in Fig. 5.21. The frequency window function $\{H(k)\}_{k=0}^{N-1}$ representing magnitude frequency response $\{|H(e^{j\omega k})|\}_{k=0}^{N-1}$ for $\omega_k = k.2\pi/N$ can stand for any type of digital filter and to evaluate real signal $\{y(n)\}_{n=0}^{N-1}$ from a real signal $\{x(n)\}_{n=0}^{N-1}$ it must keep the symmetry properties of the DFT. It is obvious that it is necessary to use sequence of values with $H(k) = H(N-k)$ for $k = 1, 2, \cdots, N-1$.

Example 5.13 *Apply the frequency domain signal processing for sequence*

$$x(n) = \sin(0.1n) + \sin(1.3n) + \sin(2n)$$

where $n = 0, 1, \cdots, N-1$ *and* $N = 64$ *using the frequency window function*

$$H(k) = \begin{cases} 1 & \textit{for } k = 0, 1, \cdots, 10 \textit{ and } k = 54, 55, \cdots, 63 \\ 0 & \textit{for } k = 11, \cdots, 53 \end{cases}$$

representing the low-pass filter with cutoff frequency $\omega_c = 10(2\pi/N) \doteq 0.98$.

Solution: Using Algorithm 5.5 with slightly modified sketch of results it is possible to obtain solution presented in Fig. 5.21 demonstrating that the lowest frequency component is preserved only.

Algorithm 5.5 *Frequency domain processing of signal* $\mathbf{x} = [x(1), x(2), \cdots, x(N)]$ *for a window function* $\mathbf{h} = [h(1), h(2), \cdots, h(N)]$.

- definition of vectors $\mathbf{x}$ and $\mathbf{h}$
- transformation to the frequency domain: $\mathbf{X}$ =fft($\mathbf{x}$)
- spectrum modification: $\mathbf{Y} = \mathbf{X}.*\mathbf{h}$
- transformation to the time domain: $\mathbf{y}$ =ifft($\mathbf{Y}$)
- sketch of results
  subplot(221);
  plot($\mathbf{x}$);plot($\mathbf{y}$) % plot of given sequences
  $\mathbf{f} = (0:N-1)/N$;plot($\mathbf{f}$,real($\mathbf{X}$), $\mathbf{f}$, $\mathbf{h}$) % frequency characteristics
  plot($\mathbf{f}$,real($\mathbf{Y}$))

5.6 Space-Scale Filtering

5.6.1 Wavelet Coefficients Thresholding

5.6.2 Signal and Image Denoising

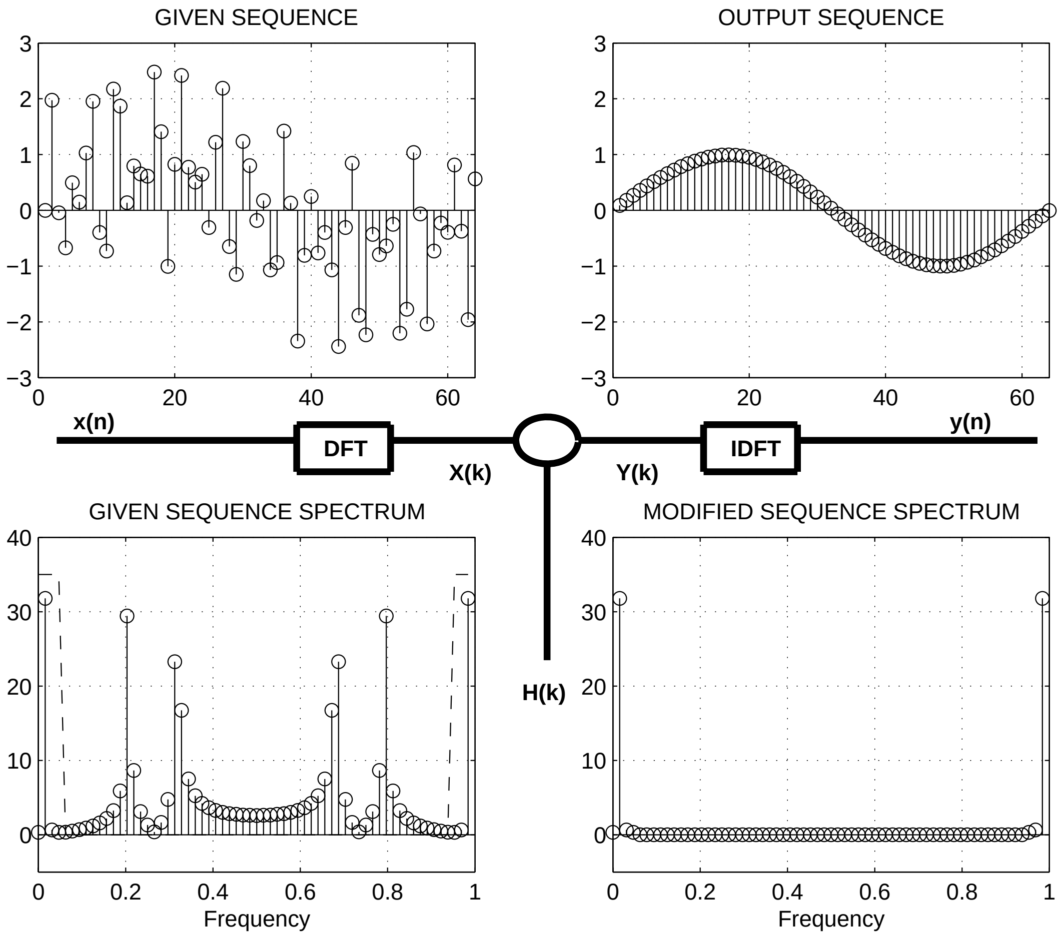

FIGURE 5.21. Principle of the frequency domain signal processing and its application for the low-pass filtering of signal $x(n) = \sin(0.1n) + \sin(1.3n) + \sin(2n)$, $n = 0, 1, \cdots, N-1$ for $N = 64$.

5.7 Summary

Digital filters are used in many engineering applications for non-adaptive and adaptive signal processing enabling signal modification, its analysis, system identification and process control as well. This chapter presented some *basic principles* and methods only for signal processing in time and frequency domain.

The main interest has been devoted to *finite impulse response filters* owing to their not too complicated design, no problems with their stability and linear phase characteristics. The use of these systems covers many areas including *filter banks* studied in this chapter as well in connection with *multirate systems* enabling to achieve a very fast signal processing.

Description and design of *infinite impulse response filters* has been restricted to the application of the methods of the least squares for the frequency characteristic approximation and basic analog system simulation.

The application of the DFT for *frequency domain signal processing* presented a very efficient method for the signal spectrum modification.

6

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