COMPARISON OF ROBUST CONTROLLER DESIGN METHODS

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Abstract

The paper deals with the comparison of robust controller design methods for uncertain SISO systems. For demonstrate the proposed controller design methods we used robust approach (e.g., Kharitonov theory, Edge Theorem, Small Gain Theorem) and genetic algorithm. The genetic algorithm represents an optimisation procedure, where the costs function to be minimized comprises the closed-loop simulation of the control process and a selected performance index evaluation. Using this approach the parameters of the PID controller were optimised in order to become the required behaviour of the control process. The practical application is illustrated by the robust controller design for a DC motor with various parameters.

1 Introduction

For many real processes a controller design has to cope with the effect of uncertainties, which very often cause a poor performance or even instability of closed-loop systems. The reason for that is a perpetual time change of parameters (due to aging, influence of environment, working point changes *etc.*), as well as unmodelled dynamics. The former uncertainty type is denoted as the parametric uncertainty and the latter one the dynamic uncertainty. A controller ensuring closed-loop stability under both of these uncertainty types is called a robust controller. A lot of robust controller design methods are known from the literature [1], [2] in the time- as well as in the frequency domains.

In this paper three approaches to robust controller design have been applied for three working points of a DC-motor with various parameters. The first method is based on the Kharitonov systems and Bode diagram for interval model. This controller guarantees the required phase margin. The second approach is accomplished with the Edge Theorem and the Neymark D-partition method for the affine model. The last controller design method is based on the Small Gain Theorem considering uncertain system model with additive uncertainty.

The identified models are used for simulation of performance index in genetic algorithm. Control performance indices corresponding to robust controllers designed for three required closed-loop stability degree are compared in three working points.

2 The laboratory model of a DC-motor

The laboratory model of a DC-motor, shown in Figure 1, consists of two co-operating real DC servomotors, where the first one is connected as a motor and the other one as a generator.

Figure 1: The laboratory model of DC motor

In our paper manipulated variable is input voltage of DC motor and controlled variable is speed but we used only output voltage of generator. The mechanical interconnection is realised with inertia load and spring. Power supply, the signals measurement and motor control is supplied by motor electronics. In electronics, there is a RC component connected to the input of motor to enable the change of time constant and gain of controlled system.

The focus of this paper is to show robust PID controller design to control angular velocity for selected area of working points. The identified models are used for the design of robust controllers and simulation of performance index in the genetic algorithm. Control performance indices corresponding to robust controllers. The performance of designed PID robust controllers are compared on the laboratory model of the DC motor in a few working points.

3 Robust controller design

3.1 The Kharitonov systems

Consider three working points of the controlled system. Comparing coefficients of equal powers of s of the three transfer functions we obtain intervals of coefficients:

$$b_i \in \left\langle \underline{b}_i, \overline{b}_i \right\rangle \quad \text{and} \quad a_j \in \left\langle \underline{a}_j, \overline{a}_j \right\rangle \quad \text{for } i = 0,1,m;\ j = 1,2,...,n \tag{1}$$

where $a_n \neq 0; b_m \neq 0; m \leq n$.

Interval polynomials of the system numerator and denominator respectively are given

$$\mathbf{N}(s) = \left\{ B(s) : b_0 + ... + b_m s^m, b_i \in \left\langle \underline{b}_i, \overline{b}_i \right\rangle, i = 0,1,...,m \right\} \tag{2}$$

$$\mathbf{D}(s) = \left\{ A(s) : a_0 + ... + a_n s^n, a_i \in \left\langle \underline{a}_i, \overline{a}_i \right\rangle, i = 0,1,...,n \right\} \tag{3}$$

The interval system is defined in the form

$$\mathbf{G}(s) = \left\{ \frac{B(s)}{A(s)} : (B(s), A(s)) \in (\mathbf{N}(s), \mathbf{D}(s)) \right\} \tag{4}$$

Consider $I(s)$ to be the set of closed-loop characteristic polynomials of degree *n* for the interval systems (4)

$$p(s) = A(s) + B(s) = p_0 + p_1 s + p_2 s^2 + ... + p_n s^n \tag{5}$$

where $p_0 \in \left\langle \underline{p}_0, \overline{p}_0 \right\rangle$, $p_1 \in \left\langle \underline{p}_1, \overline{p}_1 \right\rangle$, ..., $p_n \in \left\langle \underline{p}_n, \overline{p}_n \right\rangle$.

Such a set of polynomials is called *interval family* and we refer to *I*(*s*) as to interval polynomial. The necessary and sufficient condition for the stability of the entire family is formulated in the Kharitonov's Theorem.

Theorem 1 (Kharitonov's Theorem)

Every polynomial in the family $I(s)$ is stable if and only if the following four extreme polynomials are stable:

$$K^1(s) = \underline{p}_0 + \underline{p}_1 s + \overline{p}_2 s^2 + \overline{p}_3 s^3 + \underline{p}_4 s^4 + ... = p^{--} \tag{6}$$

$$K^2(s) = \underline{p}_0 + \overline{p}_1 s + \overline{p}_2 s^2 + \underline{p}_3 s^3 + \underline{p}_4 s^4 + ... = p^{-+}$$

$$K^3(s) = \overline{p}_0 + \underline{p}_1 s + \underline{p}_2 s^2 + \overline{p}_3 s^3 + \overline{p}_4 s^4 + ... = p^{+-}$$

$$K^4(s) = \overline{p}_0 + \overline{p}_1 s + \underline{p}_2 s^2 + \underline{p}_3 s^3 + \overline{p}_4 s^4 + ... = p^{++}$$

If the coefficients p_i vary dependently, the Kharitonov's Theorem is conservative. In such a case we can use the set of Kharitonov systems as follows:

$$\mathbf{G}_K(s) = \left\{ \frac{K_B^i(s)}{K_A^j(s)} : i, j = 1,2,3,4 \right\} \tag{7}$$

where $K_B^i(s)$, i = 1,2,3,4 and $K_A^j(s)$, j = 1,2,3,4 denote the Kharitonov polynomials associated with $\mathbf{N}(s)$ and $\mathbf{D}(s)$, respectively.

Theorem 2

The closed loop system containing the interval plant **G**(s) is robustly stable if and only if each of the Kharitonov systems in $\mathbf{G}_K(s)$ is stable.

It means that the controller is to be designed for the worst Kharitonov systems (with the least phase margin). Then standard methods can applied to the controller design, e.g. Bode characteristics with guaranteeing the required phase margin.

3.2 Robust controller design using the Edge Theorem

If a part of coefficients of the plant vary dependently, then it is better to use the affine model of the plant in the form:

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_0(s) + \sum_{i=1}^{p} b_i(s) q_i}{a_0(s) + \sum_{i=1}^{p} a_i(s) q_i} \tag{8}$$

where $q_i \in \langle \underline{q}_i, \overline{q}_i \rangle$ are uncertain coefficients. The coefficients depend linearly on uncertain parameter vector $\mathbf{q}^T = [q_1, ..., q_p]$. The parameters q_i vary within a p - dimensional box

$$\mathbf{Q} = \{ \mathbf{q} : q_i \in \langle \underline{q}_i, \overline{q}_i \rangle, i = 1, ..., p \}. \tag{9}$$

If we vary parameters $q_i = \underline{q}_i$ or $q_i = \overline{q}_i$ then is possible to obtain 2^p transfer functions with constant coefficients; inserting them to the vertices of a p - dimensional polytope, the transfer function (8) describes a so-called *polytopic system*. Consider the controller transfer function in the form

$$G_R(s) = \frac{F_1(s)}{F_2(s)} \tag{10}$$

where $F_1(s)$ and $F_2(s)$ are polynomials with constant coefficients. Then the characteristic polynomials with the polytopic system are

$$p(s, \mathbf{q}) = b_0(s) F_1(s) + a_0(s) F_2(s) + \sum_{i=1}^{p} q_i [b_i(s) F_1(s) + a_i(s) F_2(s)] \tag{11}$$

or in a more general form

$$p(s, \mathbf{q}) = p_0(s) + \sum_{i=1}^{p} q_i p_i(s) \quad q_i \in \mathbf{Q} \tag{12}$$

Theorem 3 (Edge Theorem)

The polynomial family (12) is stable if and only if the edges of **Q** are stable.

The simple stability analysis method for families of polynomials (edges of **Q**) is given in the following theorem.

Theorem 4 (Bialas)

Let $H_n^{(a)}$ and $H_n^{(b)}$ be the Hurwitz matrices of

$$\begin{aligned} p_b(s) &= p_{b0} + p_{b1}s + p_{b2}s^2 + \ldots + p_{bn}s^n \quad p_{bn} > 0, \\ p_a(s) &= p_{a0} + p_{a1}s + p_{a2}s^2 + \ldots + p_{an}s^n \quad p_{an} > 0, \end{aligned} \tag{13}$$

respectively. The polynomial family

$$p(s,Q) = \{\lambda p_a(s) + (1-\lambda)p_b(s), \quad \lambda \in [0,1]\} \tag{14}$$

is stable if and only if:

1) $p_b(s)$ is stable
2) the matrix $\left(\mathbf{H}_n^{(b)}\right)^{-1}\mathbf{H}_n^{(a)}$ has no nonpositive real eigenvalues.

Using the Edge Theorem, the controller design has to be applied to 4 vertices of the polytopic system; by applying e.g. the Neymark's D-partition method guaranteeing the required closed-loop stability degree we choose the controller coefficients such that the vertices of polytopic system are stable. Then we have to check stability of each edge of the box **Q** by e.g. the Bialas Theorem. If any of the edges is unstable, new controller coefficients are to be chosen by Neymark's method.

3.3 Robust controller design using the Small Gain Theorem

Consider a perturbed plant with unstructured additive uncertainties in the form

$$G_p(s) = G_{nom}(s) + \partial G(s) \tag{15}$$

where $G_{nom}(s)$ is the nominal model and $\partial G(s)$ are additive uncertainties.

The nominal model can be obtained e.g. by *N* identifications of the plant (in *N* working points) by taking mean values of the nominator and denominator coefficients, respectively:

$$G_{nom}(s) = \frac{(B_1(s) + \ldots + B_N(s))/N}{(A_1(s) + \ldots + A_N(s))/N} \tag{16}$$

For each ω the uncertainties are found by substituting $s = j\omega - \alpha$ where α is the required stability degree:

$$\delta G(\omega) = \max \left| G_{nom}(s) - G_{p_i}(s) \right|_{s = j\omega - \alpha} \qquad \text{for } i = 1, \ldots, N \tag{17}$$

Theorem 5 (Small Gain Theorem)

Assume that the open-loop system is stable. The closed-loop system is stable if and only if the open-loop magnitude satisfies

$$\left|G_R(j\omega)G_p(j\omega)\right| < 1 \qquad \text{for } \omega \in \langle 0, \infty\rangle \tag{18}$$

Theorem 6

Consider an auxiliary characteristic polynomial in the form

$$1 + F_{URO}(s)\frac{\delta G(s)}{G_{nom}(s)} \tag{19}$$

where
$$F_{URO}(s) = \frac{G_{nom}(s)G_R(s)}{1 + G_{nom}(s)G_R(s)}. \tag{20}$$

Assume that the open-loop system (nominal model and controller) and the auxiliary characteristic polynomial (19) are stable. Then closed-loop characteristic polynomial $p(s) = 1 + G_p(s)G_R(s)$ with unstructured additive uncertainties (15) is stable if and only if the following condition holds:

$$\left|F_{URO}(j\omega - \alpha)\right| < \frac{1}{\left|\dfrac{\delta G(\omega)}{G_{nom}(j\omega - \alpha)}\right|} = M_0(\omega) \quad \text{for } \omega \in \langle 0, \infty\rangle \tag{21}$$

The condition (21) is verified graphically. The robust controller design using Small Gain Theorem is realized according to the following steps:

1. Specify the closed – loop system magnitude corresponding to the transfer function:

$$W(s) = \frac{G_{nom}(s)G_R(s)}{1 + G_{nom}(s)G_R(s)} \tag{22}$$

   If the nominal model is of the second order then $W(s) = \dfrac{as+1}{bs+1}$

   and $G_R(s) = \dfrac{W(s)}{G_{nom}(s) - W(s)G_{nom}(s)}$ is a PID controller.

2. Choose the numerator of $W(s)$ equal to the numerator of G_{nom}
3. Choose $b > a$ and design the robust controller so that (21) is satisfied.

3.4 Genetic algorithm

As mentioned above, the aim of the control design is to provide required static and dynamic behaviour of the controlled process. Usually, this behaviour is represented in terms of the well-known concepts referred in the literature: maximum overshoot, settling time, decay rate, steady state error or various integral performance indices [3, 8].

Without loss of generality let us consider a feedback control loop (closed-loop) (Fig.2), where angular velocity *y* is the regulated variable, input voltage *u* is the manipulated variable, *w* is the reference variable of angular velocity and *e* is the control error (*e*=*w*-*y*).

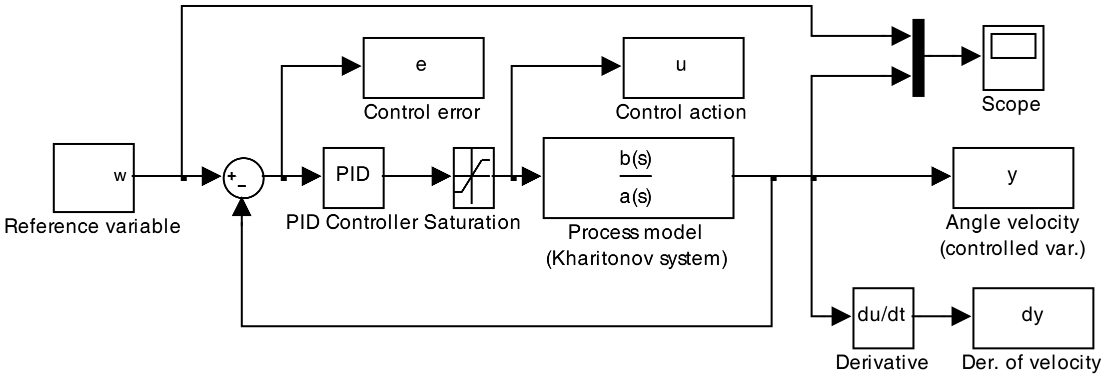

Figure 2: The control loop simulation scheme

The controller design principle is actually an optimization task - search for such controller parameters from the defined parameter space, which minimize the performance index. The cost function (fitness) is a mapping $R^n \rightarrow R$, where n is the number of designed controller parameters. The cost function can to represent sum of absolute control errors (SAE) in following forms:

$$J = \sum_{i=1}^{N} |e_i| \quad = \quad \sum_{i=1}^{N} |w_i - y_i|, \tag{23}$$

where w is reference variable, y is controlled output, e is control error and N is number of patterns. Fitness is represented by the cost function or in the case of control, by the modified cost function, which can be penalized for example by derivation of process output y or by overshoot of process output or by derivation of control action u.

The evaluation of the cost function consists of two steps. The first step is the computer simulation of the closed-loop time-response, and the second one is the performance index evaluation.

Genetic algorithms are described in e.g. [3, 4, 5, 6, 7, 8] and others. Each chromosome represents a potential solution, which is a linear string of numbers, whose items (genes) represent in our case the designed controller parameters. Because the controller parameters are real-number variables and in case of complex problems the number of the searched parameters can be large, real-coded chromosomes have been used.

The searched PID controller parameters are $K \in R^+$, $Ti \in R^+$, $Td \in R^+$. The chromosome representation in this case can be in form ***ch***$=\{K, Ti, Td\}$.

A general scheme of a GA can be described by following steps (Figure 3):

1. Initialisation of the population of chromosomes (set of randomly generated chromosomes).

2. Evaluation of the cost function (fitness) for all chromosomes.

3. Selection of parent chromosomes.

4. Crossover and mutation of the parents $\rightarrow$ children.

5. Completion of the new population from the new children and selected members of the old population. Jump to the step 2.

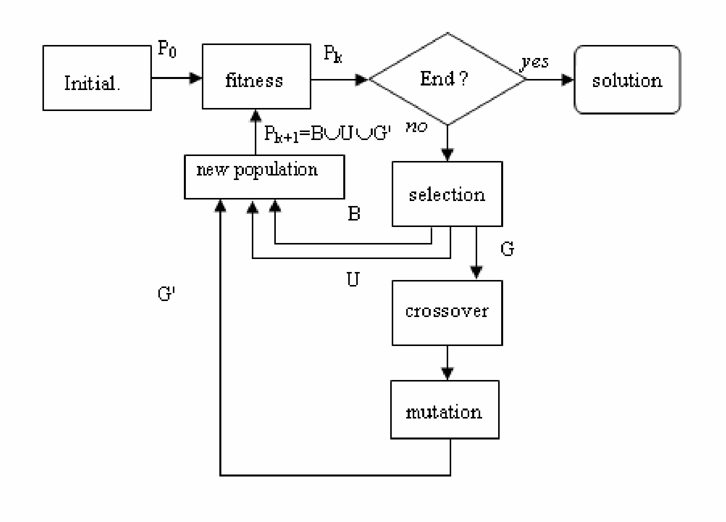

Figure 3: Block scheme of the used genetic algorithm

A block scheme of a GA-based design is in Figure 4. Before each cost function evaluation, the corresponding chromosome (genotype) is decoded into controller parameters of the simulation model (phenotype) and after the simulation the performance index is evaluated.

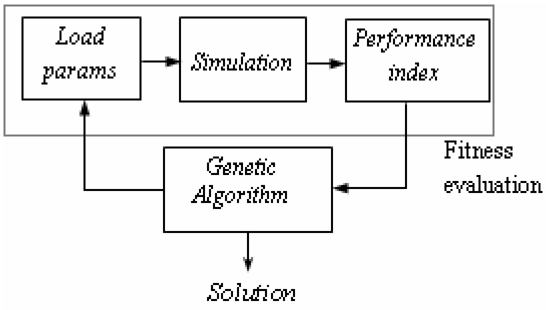

Figure 4: Block scheme of the GA-based controller design

4 Design of robust controller for the laboratory model of a DC-motor

Consider the transfer functions of a DC motor obtained by identification in three working points:

WP1: manipulated variable $u = 2$ [V]; controlled variable $y = 1.93$ [V];

load $z = 2$ [V]

$$G_{p_1}(s) = \frac{-0.06161s + 1.292}{0.3144s^2 + 0.9261s + 1}$$

WP2: $u = 2$ [V]; $y = 1.11$ [V]; $z = 2$ [V]

$$G_{p_2}(s) = \frac{0.05454s + 1.135}{0.6752s^2 + 1.619s + 1}$$

WP3: $u = 7$ [V]; $y = 7.3$ [V]; $z = 5$ [V]

$$G_{p_3}(s) = \frac{0.05925s + 1.221}{1.004s^2 + 1.805s + 1}$$

4.1 Robust controller design using Kharitonov systems

The Kharitonov approach uses the interval model:

$$G_p(s)=\frac{B(s)}{A(s)}=\frac{b_1 s+b_0}{a_2 s^2+a_1 s+1}$$

where $b_0 \in \langle 1.135, 1.292 \rangle$, $b_1 \in \langle 0.05454, 0.06161 \rangle$, $a_1 \in \langle 0.9261, 1.805 \rangle$, $a_2 \in \langle 0.3144, 1.004 \rangle$.

The required phase margin: $\Delta\varphi_z = 40[°]$. The phase added by PI controller: $\varphi_r = 45[°]$. We have designed a robust PI controller for the worst Kharitonov system (with the least phase margin) in the form

$$R(s)=K\left(1+\frac{1}{T_i s}\right)$$

where the gain $K = 0.54511$ and the integration time constant $T_i = 2.2667[s]$.

4.2 Robust controller design using the Edge Theorem

The Edge Theorem based approach uses the polytopic model:

$$G_p(s)=\frac{b_0(s)+b_1(s)q_1+b_2(s)q_2}{a_0(s)+a_1(s)q_1+a_2(s)q_2}$$

where: $b_0(s)=0.06043s+1.2565$, $\quad a_0(s)=0.6592s^2+1.3656s+1$,

$b_1(s)=0.002355s+0.043$, $\quad a_1(s)=0.1644s^2+0.093$,

$b_2(s)=-0.003535s-0.0785$, $\quad a_2(s)=0.1804s^2+0.34645s$

q_i - uncertain coefficients

The required degree of stability: $\alpha = 0$

The robust PID controller has been designed by Neymark's D-partition method for 4 vertices of the polytopic system

$$G_R(s)=K\left(1+\frac{1}{T_i s}+T_d s\right)$$

where the gain $K = 0.7$, the integration time constant $T_i = 0.63636\ [s]$ and the derivative time constant $T_d = 0.07143\ [s]$. Stability has been verified for each edge of the box **Q** by Bialas Theorem and all eigenvalues of the Bialas matrices were not nonpositive real. Therefore, the closed-loop with polytopic systems and robust controller is stable and the achieved degree in 4 vertices is $\alpha = 0.334$.

4.3 Robust controller design using the Small Gain Theorem

The Small Gain Theorem based approach uses the uncertain model with additive uncertainties. The nominal model: $G_{nom}=\dfrac{0.058075s+1.2135}{0.6592s^2+1.3656s+1}$

and uncertainties for the required degree of $\alpha = 0$ are depicted in Fig. 5. The desired transfer function: $W(s)=\dfrac{-0.0989s+1}{bs+1}$. We have designed robust PID controller:

$b = 1{,}5 \qquad K = 0.7749$, $T_i = 1.3655[s]$, $T_d = 0.4827[s]$

Condition (21) was satisfied.

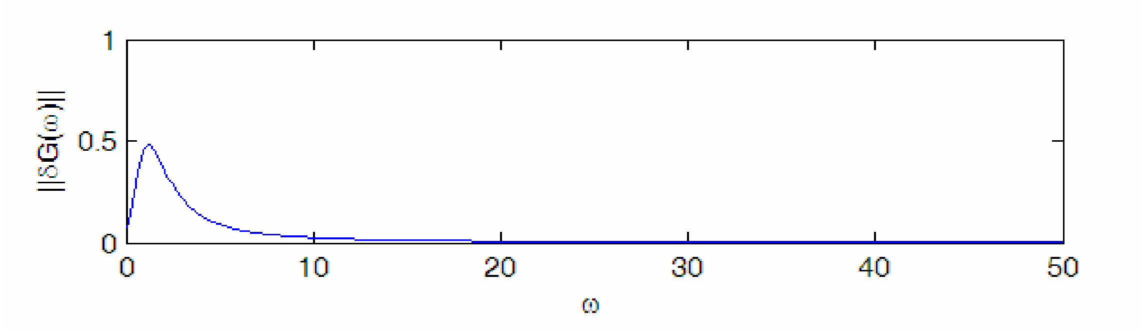

Figure 5: $\delta G(\omega)$ - versus – ω plot

4.4 Design of robust PID controller using genetic algorithm

We have designed a robust PID controller for the all four Kharitonov systems in the form

$$G_R(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right) \tag{24}$$

where the gain $K = 1.3561$ and the integration time constant $T_i = 1.2945[s]$ and the derivation time constant $T_d = 0.0535[s]$. The second design has the result:

$$K = 1.1062\,,\ T_i = 0.6067[s] \text{ and } T_d = 0.0597[s]$$

For optimization of PID controller parameters we used of genetic algorithm depictured in Figure 3. GA contained 30 chromosomes, tournament selection, one point crossover and additive mutation with mutation rate 0.2. For searching optimal PID controller parameters the used cost function (fitness) was considered by equation (23), which is penalize by overshoot of process output and derivation of control action. The fitness is evaluated in all four Kharitonov systems. In Figure 6 the cost function convergence during GA-runs (cost function vs. generation number) is depicted.

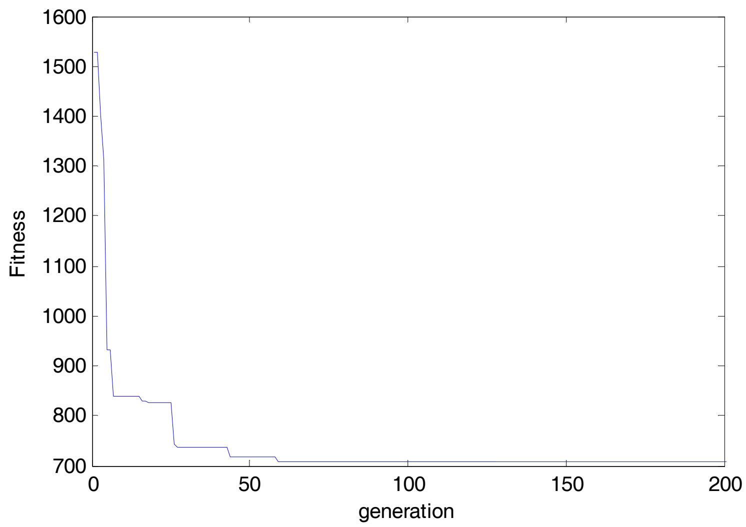

Figure 6: Evolution of the fitness function

4.5 Verification of robust controller for the laboratory model of a DC-motor

Figures 7, 8 and 9 show the step responses designed robust controller in three working points.

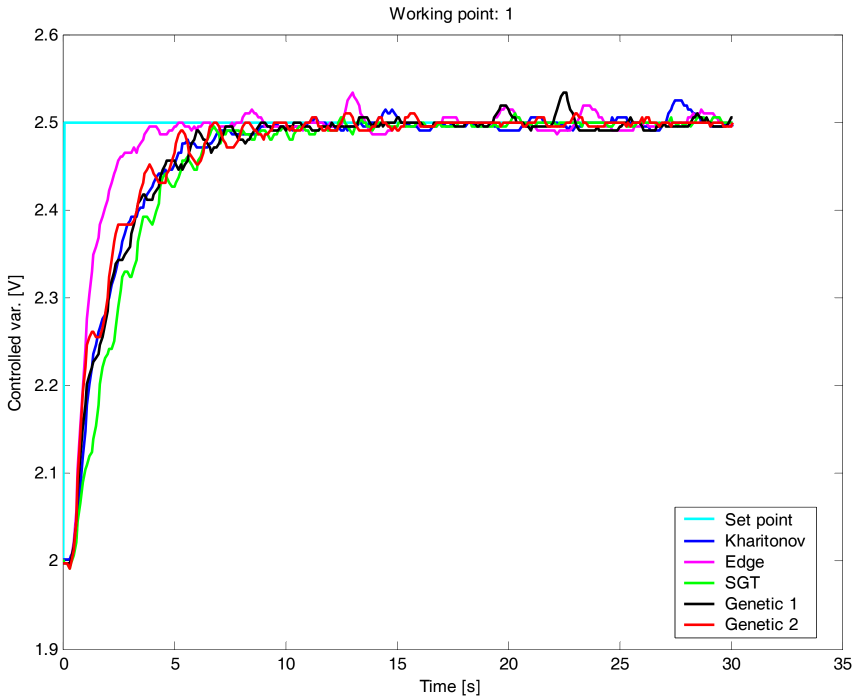

Figure 7: Step responses in the first working point

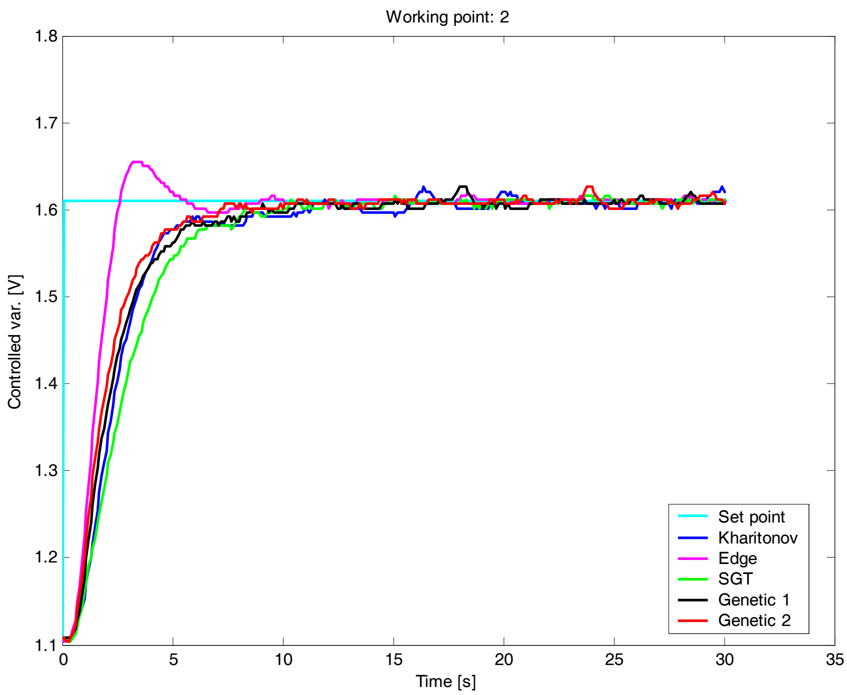

Figure 8: Step responses in the second working point

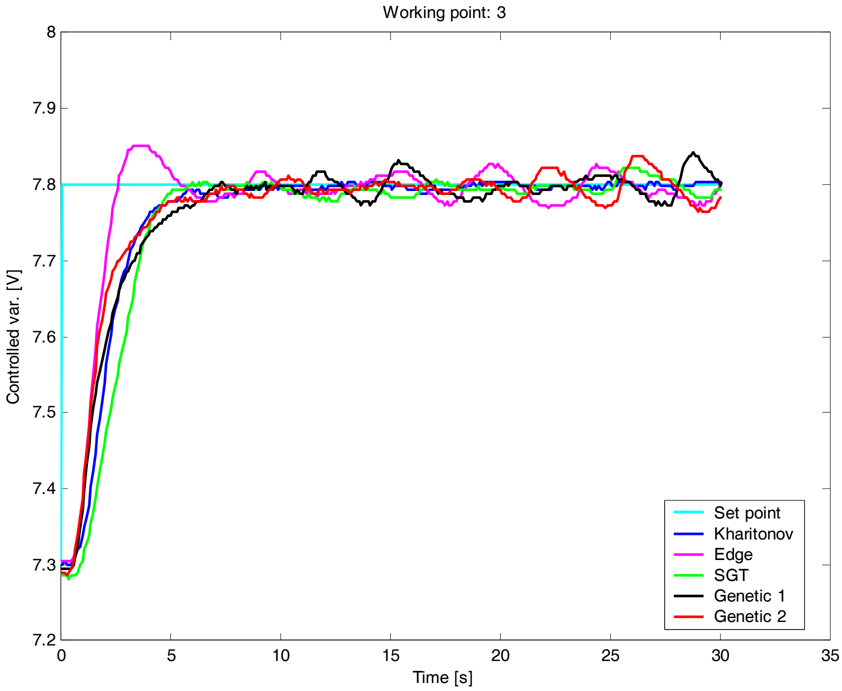

Figure 9: Step responses in the third working point

5 Conclusion

The main aim of this paper has been to apply three methods to design a robust controller and genetic algorithm for a laboratory model of a DC motor. The presented methods are based on the robust stability analysis. The first method is based on the results of Kharitonov systems and guarantees robust stability of interval system which was defined by three working points. The second approach was applied by the Edge Theorem and the Neymark D-partition method and the next method was applied by Small Gain Theorem. Using genetic algorithm the parameters of the PID controller were optimised in order to become the required behaviour of the control process. The results obtained by verification on DC motor with various parameters show the effectiveness of the proposed methods.

Acknowledgments

The work has been supported by the grants agency VEGA no. 1/0544/09 and no. 1/0690/09.

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